Numerical Analysis Preliminary Exam
9 AM TO 12 NOON, JANUARY 5, 2022

INSTRUCTIONS. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your Student ID on your exam; **do not write your name on your exam**.

Problem 1: Rootfinding

Consider a 2D fixed point iteration of the form

$$\begin{aligned} x_{k+1} &= f(x_k, y_k) \\ y_{k+1} &= g(x_k, y_k) \end{aligned}$$

Assume that the vector-valued function $\vec{h}(x, y) = (f(x, y), g(x, y))^T$ is continuously-differentiable, and the infinity norm of the Jacobian matrix is less than 1 at a unique fixed point (x_∞, y_∞).

Now consider the "nonlinear Gauss-Seidel" version of the iteration:

$$\begin{aligned} x_{k+1} &= f(x_k, y_k) \\ y_{k+1} &= g(x_{k+1}, y_k) \end{aligned}$$

Prove that the "nonlinear Gauss-Seidel" version is convergent, to the same fixed point, for initial conditions sufficiently close to the fixed point.

Problem 2: Interpolation & Approximation

Let the ordinary Legendre polynomial of degree k be denoted $P_k(x)$ for $k \geq 0$. The *associated* Legendre polynomials are

$$P_k^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_k(x),\ m > 0, k \geq m.$$

(Note that despite the name, for odd m they are not actually polynomials.)

(a) Consider the *interpolation* problem of finding coefficients a_k such that

$$\sum_{k=1}^{N} a_k P_k^1(x_i) = y_i, \quad i = 1, \ldots, N.$$

Prove that this linear system of equations for the unknown coefficients a_k is nonsingular whenever the set of interpolation points x_i does not include ± 1, and does not include duplicates.

(b) Consider the *approximation* problem of finding coefficients a_k to minimize the squared approximation error

$$\left\| f(x) - \sum_{k=1}^{N} a_k P_k^1(x) \right\|_2^2$$

where the L^2 norm is over $x \in [-1, 1]$. Write down a linear system for the unknown coefficients a_k and explain why it is nonsingular. You should give an explicit integral expression for the entries of the coefficient matrix and right hand side, but the expression does not need to be simplified.

(c) Let $\mathbf{M}$ be the coefficient matrix from (b). Prove that $\mathbf{M}_{k,j} = 0$ when $k + j$ is odd.

Problem 3: Quadrature

The quadrature formula

$$\int_0^h f(x)\mathrm{d}x \approx \frac{3h}{4} f\left(\frac{h}{3}\right) + \frac{h}{4} f(h)$$

integrates polynomials of degree ≤ 2 exactly. Derive an error bound of the form Ch^4 for this quadrature rule, where C is a constant independent of h, assuming that $f \in C^3[0, h]$. Hint: Use the Peano kernel theorem.

Problem 4: Numerical Linear Algebra

For all parts of this problem, assume all matrices and vectors are real.

(a) Write down the steepest descent method for solving $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is symmetric and positive definite.

(b) Explain how the formulas from (a) can break down if $\mathbf{A}$ is symmetric, but only non-negative definite (also called positive semi-definite).

(c) Suppose that $\mathbf{A}$ is symmetric but only non-negative definite. Show that if $\mathbf{b}$ is in the range of $\mathbf{A}$, then the steepest-descent method will still converge to a solution. You may assume that the steepest-descent method converges whenever $\mathbf{A}$ is symmetric and positive definite.

Problem 5: Ordinary Differential Equations

Consider a system of two ODEs of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y).$$

Suppose that it is more computationally expensive to evaluate g than to evaluate f.

(a) Prove that the multi-rate explicit Euler method defined by

$$\begin{aligned}
x_{k+1/2} &= x_k + \frac{\Delta}{2} f(x_k, y_k) \\
x_{k+1} &= x_{k+1/2} + \frac{\Delta}{2} f(x_{k+1/2}, y_k) \\
y_{k+1} &= y_k + \Delta g(x_k, y_k)
\end{aligned}$$

is locally second order, where Δ is the size of the time step. (Only consider the order at integer time subscripts.)

(b) Consider applying the method from (a) to the following linear problem:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -y.$$

Under what conditions on the time step $\Delta > 0$ will the discrete solution remain stable, i.e. satisfying $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = 0$ for any initial condition?

Problem 6: Partial Differential Equations

Consider following discretization

$$u_j^{n+1} = u_j^n + \frac{\kappa \Delta t}{\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

of the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

on a periodic 1D domain. Assume that the spatial grid is equispaced with size Δx.

(a) Show that the discretization is first-order accurate in time and second-order accurate in space.

(b) Show that if the time step is chosen to be $\Delta t = \Delta x^2/(6\kappa)$, then the discretization becomes second-order accurate in time.