Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August 2020

Instructions. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your student ID number (not your name!) on your exam.

Problem 1: Root finding

Consider applying Newton's method to find a root of a real cubic polynomial $f(x)$.
(a) Give a heuristic (e.g. geometric) argument showing that if two roots coincide, there is precisely one starting guess x_0 (other than the double root) for which Newton will fail, and that this point separates the basins of attraction for the distinct roots.
(b) Suppose $x = \alpha$ is a root of multiplicity 2. Prove that if Newton's method converges to α, the convergence is first order.
(c) Propose a technique for recovering second order convergence of Newton's method for finding the double root α. Prove that your proposed method is in fact second order convergent.

Solution: **(a)** The point x_0 where Newton's method will fail is the location where $f'(x) = 0$ and is not a root. Let $f(x) = (x - \alpha)^2(x - \beta)$ for α, $\beta \in \mathbb{R}$. Then $f'(x) = 2(x - \alpha)(x - \beta) + (x - \alpha)^2$. This is $x_0 = \frac{2\beta+\alpha}{3}$.
Figure 1 provides an illustration of where the point x_0 is for a specific $f(x)$.

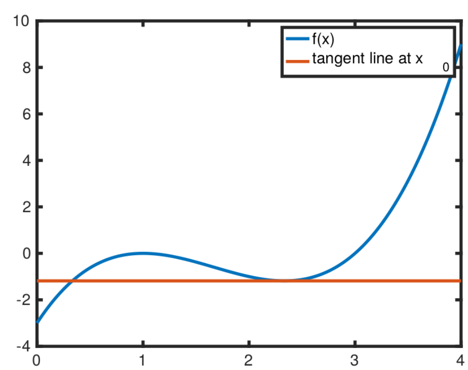

Figure 1: Illustration of x_0 for a specific $f(x)$.

(b) Let $f(x) = (x-\alpha)^2(x-\beta)$. Our goal is to show that Newton's method has first order convergence to α; i.e.

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} = C$$

for some finite constant C.

$$
\begin{aligned}
\lim_{n\to\infty}\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|} &= \lim_{n\to\infty}\frac{|x_n+\frac{f(x_n)}{f'(x_n)}-\alpha|}{|x_n-\alpha|} \\
&= \lim_{n\to\infty}\frac{|x_n+\frac{(x_n-\alpha)^2(x_n-\beta)}{2(x_n-\alpha)(x_n-\beta)+(x_-\alpha)^2}-\alpha|}{|x_n-\alpha|} \\
&= \lim_{n\to\infty}\frac{|x_n+\frac{(x_n-\alpha)(x_n-\beta)}{2(x_n-\beta)+(x_n-\alpha)}-\alpha|}{|x_n-\alpha|} \\
&= 1+\lim_{n\to\infty}\left|\frac{(x_n-\beta)}{2(x_n-\beta)+(x_n-\alpha)}\right| \\
&= 3/2
\end{aligned}
$$

(c) There are two options: (i) Apply Newton's method to $\mu(x)=\frac{f(x)}{f'(x)}$, (ii) Modify Newton's method.

$$x_{n+1}=x_n-2\frac{f(x)}{f'(x)}$$

If (i) is chosen:

$$\mu(x)=\frac{(x-\alpha)(x-\beta)}{(2(x-\beta)+(x-\alpha))}$$

Since α is a multiplicity 1 root of $\mu(x)$, Newton's method will quadratically converge to α.
If (ii) is chosen, then we need to show the new method is second order convergent.

$$
\begin{aligned}
|x_{n+1}-\alpha| &= \left|x_n-2\frac{(x_n-\alpha)^2(x_n-\beta)}{2(x_n-\alpha)(x_n-\beta)+(x_n-\alpha)^2}-\alpha\right| \\
&= \left|x_n-2\frac{(x_n-\alpha)(x_n-\beta)}{2(x_n-\beta)+(x_n-\alpha)}-\alpha\right| \\
&= |x_n-\alpha|\left|1-2\frac{(x_n-\beta)}{2(x_n-\beta)+(x_n-\alpha)}\right| \\
&= |x_n-\alpha|\left|\frac{2(x_n-\beta)+(x_n-\alpha)-2(x_n-\beta)}{2(x_n-\beta)+(x_n-\alpha)}\right| \\
&= |x_n-\alpha|^2\left|\frac{1}{2(x_n-\beta)+(x_n-\alpha)}\right|
\end{aligned}
$$

This means that

$$\lim_{n\to\infty}\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^2}=\lim_{n\to\infty}\left|\frac{1}{2(x_n-\beta)+(x_n-\alpha)}\right|=\frac{1}{2(\alpha-\beta)}$$

Thus the new method is second order convergent.

Problem 2: Quadrature
Heun's method for the ODE $y'(t) = f(y)$ is

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(\tilde{y}_{n+1}) = f(y_n + hk_1) \\ y_{n+1} &= y_n + \frac{h}{2}\left[k_1 + k_2\right] \end{aligned}$$

where h is the step size. The ODE can also be written as an integral equation

$$y(t) = y(0) + \int_0^t f(y(s))\mathrm{d}s$$

(a) Explain how a single step of Heun's rule can be thought of as a left-endpoint quadrature followed by an approximate Trapezoid rule quadrature.

(b) Use the quadrature error formula for the simple Trapezoid rule, together with Taylor series, to derive a bound on the error in one step of Heun's method. You may assume that $y_n = y(t_n)$, i.e. there is no error in y_n.

Solution: **(a)** The first stage of Heun's rule (which is a second order Runge Kutta method) is a left-endpoint quadrature (equivalent to explicit Euler)

$$y(t_{n+1}) \approx \tilde{y}_{n+1} = y_n + hf(y_n).$$

The simple trapezoid rule approximation of the integral over $[t_n, t_n + h]$ is

$$y(t_n + h) = y_n + \int_{t_n}^{t_n+h} f(y(s))\mathrm{d}s \approx y_n + \frac{h}{2}\left[f(y_n) + f(y(t_n + h))\right].$$

Since the exact value $y(t_n + h)$ is unknown, Heun's rule uses the approximation $y(t_n + h) \approx \tilde{y}_{n+1}$ to produce

$$y(t_n + h) \approx y_n + \frac{h}{2}\left[f(y_n) + f(\tilde{y}_{n+1})\right] = y_n + \frac{h}{2}\left[f(y_n) + f(y_n + hf(y_n))\right].$$

(b) For the error bound we need to account for the trapezoid rule error and the error associated with replacing $y(t_n + h)$ by $\tilde{y}_{n+1}$. To develop a formula for the error in $\tilde{y}_{n+1}$ we just need a Taylor expansion

$$y(t_n + h) = y_n + hf(y_n) + \frac{h^2}{2}f'(y(\xi))y'(\xi)$$

for some $\xi \in [t_n, t_n + h]$.
From this we find

$$|e_1| = |y(t_n + h) - \tilde{y}_{n+1}| \leq \frac{h^2}{2}\|f'(y(t))y'(t)\|_\infty.$$

For the simple trapezoid rule the error is

$$y(t_n + h) = y_n + \frac{h}{2}\left(f(y_n) + f(y(t_n + h))\right) - \frac{h^3}{12}\left.\frac{\mathrm{d}^2 f(y(t))}{\mathrm{d}t^2}\right|_{t=\zeta}$$

where ζ is an unknown value $\zeta \in [t_n, t_n + h]$. We use Taylor series to account for the error associated with replacing $y(t_n + h)$ by $\tilde{y}_{n+1}$:

$$\begin{aligned}
y(t_n + h) &= y_n + \frac{h}{2}\left(f(y_n) + f(\tilde{y}_{n+1} + e_1)\right) - \frac{h^3}{12} \left.\frac{\mathrm{d}^2 f(y(t))}{\mathrm{d}t^2}\right|_{t=\zeta} \\
&= y_n + \frac{h}{2}\big(f(y_n) + f(\tilde{y}_{n+1}) + e_1 f'(\psi)\big) - \frac{h^3}{12} \left.\frac{\mathrm{d}^2 f(y(t))}{\mathrm{d}t^2}\right|_{t=\zeta} \\
&= y_{n+1} + \frac{h e_1}{2} f'(\psi) - \frac{h^3}{12} \left.\frac{\mathrm{d}^2 f(y(t))}{\mathrm{d}t^2}\right|_{t=\zeta}
\end{aligned}$$

$$\Rightarrow y(t_n + h) - y_{n+1} = \frac{h e_1}{2} f'(\psi) - \frac{h^3}{12} \left.\frac{\mathrm{d}^2 f(y(t))}{\mathrm{d}t^2}\right|_{t=\zeta}.$$

where ψ is some unknown value of y between y_n and $y(t_n + h)$.
Using the bound above on $|e_1|$ and the triangle inequality we obtain

$$|y(t_n + h) - y_{n+1}| \leq h^3 \left[\frac{\|f'(y(t))y'(t)\|_\infty}{2} + \frac{1}{12}\left\|\frac{\mathrm{d}^2[f(y(t))]}{\mathrm{d}t^2}\right\|_\infty\right].$$

Since Heun's method is locally third order, it is globally second order.

Problem 3: Numerical Linear Algebra
The basic QR iteration for finding the eigenvalues of a real matrix A is

$$\mathrm{A}_{m-1} = \mathrm{Q}_{m-1}\mathrm{R}_{m-1}, \quad \mathrm{A}_m = \mathrm{R}_{m-1}\mathrm{Q}_{m-1}, \quad \mathrm{A}_0 = \mathrm{A}$$

where $\mathrm{A}_{m-1} = \mathrm{Q}_{m-1}\mathrm{R}_{m-1}$ is the QR factorization of A_{m-1}.

(a) Prove that the eigenvalues of A_m are the same as the eigenvalues of A.

(b) Explain how to construct an upper Hessenberg matrix H that has the same eigenvalues as A.

(c) Prove that if A is upper Hessenberg, then A_m is also upper Hessenberg. (You may assume that A is invertible, and that Q_m is upper Hessenberg whenever A_m is upper Hessenberg.)

Solution: (a) Since

$$\mathrm{A}_m = \mathrm{Q}_{m-1}^T \mathrm{A}_{m-1}\mathrm{Q}_{m-1}$$

A_m is similar to A_{m-1} so they must have the same eigenvalues. An inductive step proves that it is true for all $m \geq 1$.

(b) Let the first column of A be

$$\begin{pmatrix} a_{1,1} \\ \mathbf{a}_{2:n,1} \end{pmatrix}.$$

Let

$$\mathbf{u}_1 = \frac{\mathbf{a}_{2:n,1} - \|\mathbf{a}_{2:n,1}\|_2 \mathbf{e}_1}{\|\mathbf{a}_{2:n,1} - \|\mathbf{a}_{2:n,1}\|_2 \mathbf{e}_1\|_2}$$

and define

$$\mathrm{H}_1 = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathrm{I} - 2\mathbf{u}_1\mathbf{u}_1^T \end{array}\right].$$

Note that

$$\mathrm{H}_1\mathrm{A}\mathrm{H}_1^T = \left[\begin{array}{c|c} * & *^T \\ \hline \|\mathbf{a}_{2:n,1}\|_2 & \\ 0 & \mathrm{A}^{(2)} \end{array}\right].$$

If we recursively apply this idea to $\mathrm{A}^{(2)}$ using a matrix H_2 of the form

$$\mathrm{H}_2 = \left[\begin{array}{cc|c} 1 & 0 & \mathbf{0}^T \\ 0 & 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & \mathrm{I} - 2\mathbf{u}_2\mathbf{u}_2^T \end{array}\right]$$

etc., then we can see that the final result

$$\mathrm{H} = \mathrm{H}_{n-2}\cdots\mathrm{H}_1\mathrm{A}\mathrm{H}_1^T\cdots\mathrm{H}_{n-2}^T$$

will be upper Hessenberg.

Since the H_i are all orthogonal matrices this is a similarity transform, so H has the same eigenvalues as A.

(c) This is true even if A is not invertible, but the proof for invertible matrices is easier. Start with

$$A = Q_0 R_0.$$

Next note that

$$A_1 = R_0 Q_0 = R_0 A R_0^{-1}.$$

The lower bandwidths of the matrices in the rightmost expression are 0, 1, and 0. Since multiplication adds the bandwidths, the product on the right has lower bandwidth ≤ 1, meaning that it is upper Hessenberg. An inductive step shows that A_m will be upper Hessenberg for all $m \geq 1$.

Problem 4: Interpolation/Approximation

(a) Find the quadratic that interpolates the following temperature data: $T(-1) = 4$, $T(0) = 10$, $T(1) = 20$.

(b) Suppose that density is related to temperature via $\rho(T) = \rho_0 - \alpha T$, and that the total mass is known to be m

$$\int_{-1}^{1} \rho(T(x))\mathrm{d}x = m.$$

Find a cubic polynomial that interpolates the data from (a) and satisfies the above integral constraint, or explain why none exists. For part (b) let $\rho_0 = 1/2$, $m = 1$, and $\alpha = 1$.

(c) Consider the problem of both interpolating the data and satisfying the integral constraint for an arbitrary set of $n+1$ distinct interpolation nodes $x_0, \ldots, x_n$ using a polynomial of degree $\leq n+1$. When a solution exists, it must have the form

$$p(x) = q(x) + \sum_{j=0}^{n} T(x_j)\ell_j(x).$$

(i) When a solution does exist, give an explicit formula for $q(x)$.

(ii) Give an explicit criterion for when a solution does not exist.

Solution: **(a)** The solution is

$$p(x) = 10 + 8x + 2x^2.$$

(b) One way to approach this is to let

$$p(x) = \sum_{j=0}^{3} a_j x^j$$

and then solve the following system

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & \frac{2}{3} & 0 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 20 \\ -\frac{m-2\rho_0}{\alpha} \end{pmatrix}.$$

If you row reduce you will end up with a final equation of the form

$$0 = -\frac{64}{3}$$

so no solution exists.

(c)

(i) q must have the form

$$q(x) = \lambda \ell(x)$$

where $\ell(x) = \Pi_{j=0}^{n}(x - x_j)$ is the node polynomial. This is the only possible solution for a polynomial q of degree at most $n+1$ that satisfies $q(x_j) = 0$ for $j = 0, \ldots, n$.

The above is not a complete solution; we still need λ. The integral constraint is

$$\lambda \int_{-1}^{1} \ell(x)\mathrm{d}x + \sum_{j=0}^{n} T(x_j) \int_{-1}^{1} \ell_j(x)\mathrm{d}x = -\frac{m - 2\rho_0}{\alpha}.$$

The formula for λ is

$$\lambda = -\frac{1}{\int_{-1}^{1} \ell(x)\mathrm{d}x} \left[\frac{m - 2\rho_0}{\alpha} + \sum_{j=0}^{n} T(x_j) \int_{-1}^{1} \ell_j(x)\mathrm{d}x \right].$$

(ii) It is tempting to say that a solution does not exist when

$$\int_{-1}^{1} \ell(x)\mathrm{d}x = 0$$

which happened in part (b). But this is only half the answer. If, by chance, we have

$$\sum_{j=0}^{n} T(x_j) \int_{-1}^{1} \ell_j(x)\mathrm{d}x = -\frac{m - 2\rho_0}{\alpha}$$

then $\lambda = 0$ is an answer regardless of the value of $\int_{-1}^{1} \ell(x)\mathrm{d}x$. In this case there are an infinite number of solutions. So the full criterion for when a solution does not exist is

$$\int_{-1}^{1} \ell(x)\mathrm{d}x = 0 \textbf{ and } \sum_{j=0}^{n} T(x_j) \int_{-1}^{1} \ell_j(x)\mathrm{d}x \neq -\frac{m - 2\rho_0}{\alpha}.$$

Problem 5: Numerical ODE
Consider the boundary value problem

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = f(x), \quad u(0) = u(1) = 0$$

where $a(x) > \delta \geq 0$ is a bounded differentiable function in $[0,1]$. We note that the above ODE can be written as

$$-\frac{da}{dx}\frac{du}{dx} - a(x)\frac{d^2u}{dx^2} = f(x), \quad u(0) = u(1) = 0.$$

We assume that, although $a(x)$ is available, an expression for its derivative, $\frac{da}{dx}$, is not available.
(a) Using finite differences and an equally spaced grid in $[0,1]$, $x_l = hl$, $l = 0, \dots, n$ and $h = 1/n$, we discretize the ODE to obtain a linear system of equations, yielding an $O(h^2)$ approximation of the ODE. After the application of the boundary conditions, the resulting coefficient matrix of the linear system is an $(n-1) \times (n-1)$ tridiagonal matrix.
Provide a derivation and write down the resulting linear system (by giving the expressions of the elements).
(b) Utilizing all the information provided, find a disc in $\mathbb{C}$, the smaller the better, that is guaranteed to contain all the eigenvalues of the linear system constructed in part (a).

Solution:
(a) We must choose an $O(h^2)$ finite difference approximation for the derivatives. I choose to use centered differences to approximate the derivatives since it is known that

$$\left.\frac{du}{dx}\right|_{x_j} = \frac{u(x_{j+1}) - u(x_{j-1})}{2h} + O(h^2)$$

and

$$\left.\frac{d^2u}{dx^2}\right|_{x_j} = \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} + O(h^2).$$

Let u_j denote the approximate value $u(x_j)$. Then truncating the $O(h^2)$ terms of the derivatives and plugging them into the differential equation, we find the following row equation at x_j for $j = 1, \dots, n-1$

$$-\left(\frac{a(x_{j+1}) - a(x_{j-1})}{4h^2}\right)(u_{j+1} - u_{j-1}) - \frac{a(x_j)}{h^2}(u_{j+1} - 2u_j + u_{j-1}) = f(x_j) \tag{1}$$

Since we know that $u(0) = u(1) = 0$, we get different equations for $j = 1$ and $j = n-1$.
For $j = 1$, the equation is given by

$$-\left(\frac{a(x_2) - a(x_0)}{4h^2}\right)(u_2) - \frac{a(x_1)}{h^2}(u_2 - 2u_1) = f(x_1). \tag{2}$$

For $j = n-1$, the equation is given by

$$-\left(\frac{a(x_n) - a(x_{n-2})}{4h^2}\right)(-u_{n-2}) - \frac{a(x_{n-1})}{h^2}(-2u_{n-1} + u_{n-2}) = f(x_{n-1}). \tag{3}$$

The resulting tridiagonal system has RHS $f(x_j)$ for $j = 1, 2, \cdots, n-1$;
— diagonal entries:

$$2\frac{a(x_j)}{h^2}, \quad j = 1, \cdots, n-1;$$

— lower diagonal entries:

$$\left(\frac{a(x_{j+1}) - a(x_{j-1})}{4h^2}\right) - \frac{a(x_j)}{h^2} \quad j = 2, \cdots, n-1;$$

— upper diagonal entries:

$$-\left(\frac{a(x_{j+1}) - a(x_{j-1})}{4h^2}\right) - \frac{a(x_j)}{h^2} \quad j = 1, \cdots, n-2.$$

(b) The Gershgorin theorem states that for any complex $n \times n$ matrix $\mathbf{A}$, the eigenvalues lie within the collection of disc with radius $R_i = \sum_{j \neq i} |a_{ij}|$ centered at a_{ii}.
Applying this to the matrix, we find that the Gershgorin disc for the linear system are

$$\left|\lambda - \frac{2a(x_1)}{h^2}\right| \leq \left|\frac{a(x_2) - a(0)}{4h^2} + \frac{a(x_1)}{h^2}\right|,$$

$$\left|\lambda - \frac{2a(x_j)}{h^2}\right| \leq 2\left|\frac{a(x_{j+1}) - a(x_{j-1})}{4h^2} + \frac{a(x_j)}{h^2}\right|, \qquad \text{for } j = 2, \ldots, n-2,$$

and

$$\left|\lambda - \frac{2a(x_{n-1})}{h^2}\right| \leq \left|\frac{a(1) - a(x_{n-2})}{4h^2} + \frac{a(x_{n-1})}{h^2}\right|.$$

Now we must find a disc which contains all of these. To do this, we utilize the fact that $a(x) \geq \delta > 0$. Let $M = \max_{x \in [0,1]} |a(x)|$. Then

$$\left|\lambda - \frac{2a(x_j)}{h^2}\right| \leq 2\left|\frac{a(x_{j+1}) - a(x_{j-1})}{4h^2} + \frac{a(x_j)}{h^2}\right| \leq \frac{2}{h^2}\left(\frac{M - \delta}{4} + M\right).$$

Now that we have an upper bound for the radius of all the disc, we note that $\delta < a(x_j) \leq M \ \forall\ j$. Thus to make sure we capture all eigenvalues, we center the disc at $\frac{\delta + M}{h^2}$ in the middle of $\frac{2\delta}{h^2}$ and $\frac{2M}{h^2}$, then add $\frac{M-\delta}{h^2}$ to the radius we found above.
Finally the smallest disc that we find is

$$\left|\lambda - \frac{M + \delta}{h^2}\right| \leq \frac{M - \delta}{h^2} + \frac{2}{h^2}\left(\frac{M - \delta}{4} + M\right) = \frac{1}{h^2}\left(3.5M - 1.5\delta\right)$$

Problem 6: Numerical PDE

Consider the equation

$$\begin{aligned} u_t + au_x &= 0, \qquad a \in \mathbb{R},\ t > 0, \\ u(x,0) &= f(x). \end{aligned} \tag{4}$$

The solution will be approximated using the Finite Difference Lax-Wendroff method

$$v_j^{n+1} = v_j^n - \frac{a\Delta t}{2\Delta x}\left(v_{j+1}^n - v_{j-1}^n\right) + a^2\frac{(\Delta t)^2}{2(\Delta x)^2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right)$$

where $v_j^n = u(x_j, t_n)$ is a grid function, and Δx and Δt denote the spacing between grid points in the x and t directions.

NOTE: That there are two centered difference formulas used in the spatial direction.

(a)

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = u_t + \frac{\Delta t}{2}u_{tt} + \frac{(\Delta t)^2}{6}u_{ttt} + O((\Delta t)^3)$$

Rewrite this expression replacing the temporal derivatives u_{tt} and u_{ttt} in terms of spatial derivatives using equation (4).

(b) Determine the spatial and temporal orders of accuracy of the Lax-Wendroff method.

(c) Use von Neumann analysis to determine under what conditions the method is stable. (Hint: it is useful to look at the square of the amplification factor.)

Solution:

(a)

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = u_t + \frac{a^2\Delta t}{2}u_{xx} + \frac{a^3(\Delta t)^2}{6}u_{xxx} + O((\Delta t)^3)$$

(b)

$$\frac{v_{j+1}^n - v_{j-1}^{n-1}}{2\Delta x} = u_x + \frac{(\Delta x)^2}{6}u_{xxx} + O((\Delta x)^4)$$

$$\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2} = u_{xx} + \frac{(\Delta x)^2}{12}u_{xxxx} + O((\Delta x)^4)$$

Plugging these two Taylor expansions and the solution to part (a) into the finite difference scheme, we get

$$u_t + au_x = -\frac{a^3(\Delta t)^2}{6}u_{xxx} - \frac{a(\Delta x)^2}{6}u_{xxx} + O((\Delta t)^3) + O(\Delta x)^4)$$

So the method is $O((\Delta x)^2 + (\Delta t)^2)$.

(c) We plug $v_j^n = \xi^n e^{ikx_j}$ into the finite difference method.

$$\begin{aligned} v_j^{n+1} &= v_j^n - \frac{a\Delta t}{2\Delta x}\left(v_{j+1}^n - v_{j-1}^n\right) + a^2\frac{(\Delta t)^2}{2(\Delta x)^2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) \\ \xi^{n+1}e^{ikx_j} &= \xi^n e^{ikx_j} - \frac{a\Delta t}{2\Delta x}\left(\xi^n e^{ik(x_j+\Delta x)} - \xi^n e^{ik(x_j-\Delta x)}\right) + a^2\frac{(\Delta t)^2}{2(\Delta x)^2}\left(\xi^n e^{ik(x_j+\Delta x)} - 2\xi^n e^{ikx_j} + \xi^n e^{ik(x_j-\Delta x)}\right) \\ \xi &= 1 - \frac{a\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) + a^2\frac{(\Delta t)^2}{2(\Delta x)^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) \\ &= 1 - \frac{a\Delta t}{\Delta x}i\sin(k\Delta x) + a^2\frac{(\Delta t)^2}{2(\Delta x)^2}\left(\cos(k\Delta x) - 1\right) \end{aligned}$$

Let $\kappa = \frac{a\Delta t}{\Delta x}$, then ξ can be written in a slightly cleaner form.

$$\xi = 1 - \frac{a\Delta t}{\Delta x} i \sin(k\Delta x) + \kappa^2 \left(\cos(k\Delta x) - 1\right)$$

$$\begin{aligned}
|\xi|^2 &= 1 - 2\kappa^2(1 - \cos(k\Delta x)) + \kappa^2(1 - \cos^2(k\Delta x)) + \kappa^4(1 - \cos(k\Delta x))^2 \\
&= 1 - \kappa^2(1 - \kappa^2)(1 - \cos(k\Delta x))^2 \\
&= 1 - 4\kappa^2(1 - \kappa^2)\sin^4\left(\frac{1}{2}k\Delta x\right)
\end{aligned}$$

The method is stable when $|\xi|^2 \leq 1$ which matches the CFL condition requiring $|\kappa| \leq 1$.