Problem 1: Root finding

Consider applying Newton’s method to find a root of a real cubic polynomial \( f(x) \).

(a) Give a heuristic (e.g. geometric) argument showing that if two roots coincide, there is precisely one starting guess \( x_0 \) (other than the double root) for which Newton will fail, and that this point separates the basins of attraction for the distinct roots.

(b) Suppose \( x = \alpha \) is a root of multiplicity 2. Prove that if Newton’s method converges to \( \alpha \), the convergence is first order.

(c) Propose a technique for recovering second order convergence of Newton’s method for finding the double root \( \alpha \). Prove that your proposed method is in fact second order convergent.

Solution: (a) The point \( x_0 \) where Newton’s method will fail is the location where \( f'(x) = 0 \) and is not a root. Let \( f(x) = (x - \alpha)^2(x - \beta) \) for \( \alpha, \beta \in \mathbb{R} \). Then \( f'(x) = 2(x - \alpha)(x - \beta) + (x - \alpha)^2 \).

This is \( x_0 = \frac{2\beta + \alpha}{3} \).

Figure 1 provides an illustration of where the point \( x_0 \) is for a specific \( f(x) \).

(b) Let \( f(x) = (x - \alpha)^2(x - \beta) \). Our goal is to show that Newton’s method has first order convergence to \( \alpha \); i.e.

\[
\lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} = C
\]

for some finite constant \( C \).
\[
\lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|} = \lim_{n \to \infty} \frac{|x_n + \frac{f(x_n)}{f'(x_n)} - \alpha|}{|x_n - \alpha|} \\
= \lim_{n \to \infty} \frac{|x_n + \frac{(x_n - \alpha)(x_n - \beta)}{2(x_n - \beta) + (x_n - \beta)^2} - \alpha|}{|x_n - \alpha|} \\
= \lim_{n \to \infty} \frac{|x_n + \frac{(x_n - \alpha)(x_n - \beta)}{2(x_n - \beta) + (x_n - \alpha)} - \alpha|}{|x_n - \alpha|} \\
= 1 + \lim_{n \to \infty} \frac{|x_n - \beta|}{2(x_n - \beta) + (x_n - \alpha)} \\
= \frac{3}{2}
\]

(c) There are two options: (i) Apply Newton’s method to \( \mu(x) = \frac{f(x)}{f'(x)} \), (ii) Modify Newton’s method.

\[
x_{n+1} = x_n - 2 \frac{f(x)}{f'(x)}
\]

If (i) is chosen:

\[
\mu(x) = \frac{(x - \alpha)(x - \beta)}{(2(x - \beta) + (x - \alpha))}
\]

Since \( \alpha \) is a multiplicity 1 root of \( \mu(x) \), Newton’s method will quadratically converge to \( \alpha \).

If (ii) is chosen, then we need to show the new method is second order convergent.

\[
|x_{n+1} - \alpha| = \left| x_n - 2 \frac{(x_n - \alpha)^2 (x_n - \beta)}{2(x_n - \alpha)(x_n - \beta) + (x_n - \alpha)^2} - \alpha \right| \\
= |x_n - \alpha| \left| 1 - 2 \frac{(x_n - \beta)}{2(x_n - \beta) + (x_n - \alpha)} \right| \\
= |x_n - \alpha|^2 \left| \frac{1}{2(x_n - \beta) + (x_n - \alpha)} \right|
\]

This means that

\[
\lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^2} = \lim_{n \to \infty} \frac{1}{2(x_n - \beta) + (x_n - \alpha)} = \frac{1}{2(\alpha - \beta)}
\]

Thus the new method is second order convergent.
Problem 2: Quadrature

Heun’s method for the ODE \( y'(t) = f(y) \) is

\[
\begin{align*}
    k_1 &= f(y_n) \\
    k_2 &= f(\tilde{y}_{n+1}) = f(y_n + h k_1) \\
    y_{n+1} &= y_n + \frac{h}{2} [k_1 + k_2]
\end{align*}
\]

where \( h \) is the step size. The ODE can also be written as an integral equation

\[
y(t) = y(0) + \int_0^t f(y(s)) ds
\]

(a) Explain how a single step of Heun’s rule can be thought of as a left-endpoint quadrature followed by an approximate Trapezoid rule quadrature.

(b) Use the quadrature error formula for the simple Trapezoid rule, together with Taylor series, to derive a bound on the error in one step of Heun’s method. You may assume that \( y_n = y(t_n) \), i.e. there is no error in \( y_n \).

Solution: (a) The first stage of Heun’s rule (which is a second order Runge Kutta method) is a left-endpoint quadrature (equivalent to explicit Euler)

\[
y(t_{n+1}) \approx \tilde{y}_{n+1} = y_n + h f(y_n).
\]

The simple trapezoid rule approximation of the integral over \([t_n, t_n + h]\) is

\[
y(t_n + h) = y_n + \int_{t_n}^{t_n+h} f(y(s)) ds \approx y_n + \frac{h}{2} [f(y_n) + f(y(t_n + h))].
\]

Since the exact value \( y(t_n + h) \) is unknown, Heun’s rule uses the approximation \( y(t_n + h) \approx \tilde{y}_{n+1} \) to produce

\[
y(t_n + h) \approx y_n + \frac{h}{2} [f(y_n) + f(\tilde{y}_{n+1})] = y_n + \frac{h}{2} [f(y_n) + f(y_n + h f(y_n))].
\]

(b) For the error bound we need to account for the trapezoid rule error and the error associated with replacing \( y(t_n + h) \) by \( \tilde{y}_{n+1} \). To develop a formula for the error in \( \tilde{y}_{n+1} \) we just need a Taylor expansion

\[
y(t_n + h) = y_n + h f(y_n) + \frac{h^2}{2} f'(y(\xi)) y'(\xi)
\]

for some \( \xi \in [t_n, t_n + h] \).

From this we find

\[
|e_1| = |y(t_n + h) - \tilde{y}_{n+1}| \leq \frac{h^2}{2} \|f'(y(t))y'(t)\|_\infty.
\]

For the simple trapezoid rule the error is

\[
y(t_n + h) = y_n + \frac{h}{2} (f(y_n) + f(y(t_n + h))) - \frac{h^3}{12} \frac{d^2 f(y(t))}{dt^2} \bigg|_{t=\xi}
\]
where $\zeta$ is an unknown value $\zeta \in [t_n, t_n+h]$. We use Taylor series to account for the error associated with replacing $y(t_n+h)$ by $\tilde{y}_{n+1}$:

\[
y(t_n+h) = y_n + \frac{h}{2} (f(y_n) + f(\tilde{y}_{n+1} + e_1)) - \frac{h^3}{12} \left. \frac{d^2 f(y(t))}{dt^2} \right|_{t=\zeta}
\]

\[
= y_n + \frac{h}{2} (f(y_n) + f(\tilde{y}_{n+1}) + e_1f'(\psi)) - \frac{h^3}{12} \left. \frac{d^2 f(y(t))}{dt^2} \right|_{t=\zeta}
\]

\[
= y_{n+1} + \frac{he_1}{2} f'(\psi) - \frac{h^3}{12} \left. \frac{d^2 f(y(t))}{dt^2} \right|_{t=\zeta}
\]

\[
\Rightarrow y(t_n+h) - y_{n+1} = \frac{he_1}{2} f'(\psi) - \frac{h^3}{12} \left. \frac{d^2 f(y(t))}{dt^2} \right|_{t=\zeta}.
\]

where $\psi$ is some unknown value of $y$ between $y_n$ and $y(t_n+h)$.

Using the bound above on $|e_1|$ and the triangle inequality we obtain

\[
|y(t_n+h) - y_{n+1}| \leq h^3 \left[ \frac{||f'(y(t))y'(t)||_\infty}{2} + \frac{1}{12} \left\| \frac{d^2 f(y(t))}{dt^2} \right\|_\infty \right].
\]

Since Heun’s method is locally third order, it is globally second order.
Problem 3: Numerical Linear Algebra

The basic QR iteration for finding the eigenvalues of a real matrix $A$ is

$$A_{m-1} = Q_{m-1}R_{m-1}, \quad A_m = R_{m-1}Q_{m-1}, \quad A_0 = A$$

where $A_{m-1} = Q_{m-1}R_{m-1}$ is the QR factorization of $A_{m-1}$.

(a) Prove that the eigenvalues of $A_m$ are the same as the eigenvalues of $A$.

(b) Explain how to construct an upper Hessenberg matrix $H$ that has the same eigenvalues as $A$.

(c) Prove that if $A$ is upper Hessenberg, then $A_m$ is also upper Hessenberg. (You may assume that $A$ is invertible, and that $Q_m$ is upper Hessenberg whenever $A_m$ is upper Hessenberg.)

Solution: (a) Since

$$A_m = Q_{m-1}^T A_{m-1} Q_{m-1}$$

$A_m$ is similar to $A_{m-1}$ so they must have the same eigenvalues. An inductive step proves that it is true for all $m \geq 1$.

(b) Let the first column of $A$ be

$$\begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix}.$$ 

Let

$$u_1 = \frac{a_{2,1} - \|a_{2,1}\|_2 e_1}{\|a_{2,1} - \|a_{2,1}\|_2 e_1\|_2}$$

and define

$$H_1 = \begin{bmatrix} 1 & 0^T \\ 0 & I - 2u_1u_1^T \end{bmatrix}.$$ 

Note that

$$H_1 A H_1^T = \begin{bmatrix} * & s^T \\ \|a_{2,1}\|_2 & A^{(2)} \end{bmatrix}.$$ 

If we recursively apply this idea to $A^{(2)}$ using a matrix $H_2$ of the form

$$H_2 = \begin{bmatrix} 1 & 0 & 0^T \\ 0 & 1 & 0^T \\ 0 & 0 & I - 2u_2u_2^T \end{bmatrix}$$

etc., then we can see that the final result

$$H = H_{n-2} \cdots H_1 A H_1^T \cdots H_{n-2}^T$$

will be upper Hessenberg.

Since the $H_i$ are all orthogonal matrices this is a similarity transform, so $H$ has the same eigenvalues as $A$. 

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(c) This is true even if $A$ is not invertible, but the proof for invertible matrices is easier. Start with

$$A = Q_0R_0.$$  

Next note that

$$A_1 = R_0Q_0 = R_0AR_0^{-1}.$$  

The lower bandwidths of the matrices in the rightmost expression are 0, 1, and 0. Since multiplication adds the bandwidths, the product on the right has lower bandwidth $\leq 1$, meaning that it is upper Hessenberg. An inductive step shows that $A_m$ will be upper Hessenberg for all $m \geq 1$. 
Problem 4: Interpolation/Approximation

(a) Find the quadratic that interpolates the following temperature data: $T(-1) = 4$, $T(0) = 10$, $T(1) = 20$.

(b) Suppose that density is related to temperature via $\rho(T) = \rho_0 - \alpha T$, and that the total mass is known to be $m$

$$\int_{-1}^{1} \rho(T(x))dx = m.$$ 

Find a cubic polynomial that interpolates the data from (a) and satisfies the above integral constraint, or explain why none exists. For part (b) let $\rho_0 = 1/2$, $m = 1$, and $\alpha = 1$.

(c) Consider the problem of both interpolating the data and satisfying the integral constraint for an arbitrary set of $n + 1$ distinct interpolation nodes $x_0, \ldots, x_n$ using a polynomial of degree $\leq n + 1$. When a solution exists, it must have the form

$$p(x) = q(x) + \sum_{j=0}^{n} T(x)\ell_j(x).$$

(i) When a solution does exist, give an explicit formula for $q(x)$.

(ii) Give an explicit criterion for when a solution does not exist.

Solution: (a) The solution is 

$$p(x) = 10 + 8x + 2x^2.$$ 

(b) One way to approach this is to let 

$$p(x) = \sum_{j=0}^{3} a_j x^j$$

and then solve the following system

$$\begin{bmatrix}
1 & -1 & 1 & -1 \\
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
2 & 0 & \frac{2}{3} & 0
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
10 \\
20 \\
-\frac{m-2\rho_0}{\alpha}
\end{bmatrix}.$$ 

If you row reduce you will end up with a final equation of the form 

$$0 = -\frac{64}{3}$$

so no solution exists.

(c)

(i) $q$ must have the form 

$$q(x) = \lambda \ell(x)$$

where $\ell(x) = \Pi_{j=0}^{n}(x - x_j)$ is the node polynomial. This is the only possible solution for a polynomial $q$ of degree at most $n + 1$ that satisfies $q(x_j) = 0$ for $j = 0, \ldots, n$. 

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The above is not a complete solution; we still need $\lambda$. The integral constraint is

$$\lambda \int_{-1}^{1} \ell(x)dx + \sum_{j=0}^{n} T(x_j) \int_{-1}^{1} \ell_j(x)dx = \frac{-m - 2\rho_0}{\alpha}.$$ 

The formula for $\lambda$ is

$$\lambda = -\frac{1}{\int_{-1}^{1} \ell(x)dx} \left[ \frac{m - 2\rho_0}{\alpha} + \sum_{j=0}^{n} T(x_j) \int_{-1}^{1} \ell_j(x)dx \right].$$

(ii) It is tempting to say that a solution does not exist when

$$\int_{-1}^{1} \ell(x)dx = 0$$

which happened in part (b). But this is only half the answer. If, by chance, we have

$$\sum_{j=0}^{n} T(x_j) \int_{-1}^{1} \ell_j(x)dx = \frac{-m - 2\rho_0}{\alpha}$$

then $\lambda = 0$ is an answer regardless of the value of $\int_{-1}^{1} \ell(x)dx$. In this case there are an infinite number of solutions. So the full criterion for when a solution does not exist is

$$\int_{-1}^{1} \ell(x)dx = 0 \text{ and } \sum_{j=0}^{n} T(x_j) \int_{-1}^{1} \ell_j(x)dx \neq \frac{-m - 2\rho_0}{\alpha}.$$
Problem 5: Numerical ODE
Consider the boundary value problem

$$-\frac{d}{dx} \left( a(x) \frac{du}{dx} \right) = f(x), \quad u(0) = u(1) = 0$$

where $a(x) > \delta \geq 0$ is a bounded differentiable function in $[0, 1]$. We note that the above ODE can be written as

$$\frac{da}{dx} \frac{du}{dx} - a(x) \frac{d^2u}{dx^2} = f(x), \quad u(0) = u(1) = 0.$$

We assume that, although $a(x)$ is available, an expression for its derivative, $\frac{da}{dx}$, is not available.

(a) Using finite differences and an equally spaced grid in $[0, 1]$, $x_l = hl$, $l = 0, \ldots, n$ and $h = 1/n$, we discretize the ODE to obtain a linear system of equations, yielding an $O(h^2)$ approximation of the ODE. After the application of the boundary conditions, the resulting coefficient matrix of the linear system is an $(n-1) \times (n-1)$ tridiagonal matrix.

Provide a derivation and write down the resulting linear system (by giving the expressions of the elements).

(b) Utilizing all the information provided, find a disc in $\mathbb{C}$, the smaller the better, that is guaranteed to contain all the eigenvalues of the linear system constructed in part (a).

Solution:

(a) We must choose an $O(h^2)$ finite difference approximation for the derivatives. I choose to use centered differences to approximate the derivatives since it is known that

$$\left. \frac{du}{dx} \right|_{x_j} = \frac{u(x_{j+1}) - u(x_{j-1})}{2h} + O(h^2)$$

and

$$\left. \frac{d^2u}{dx^2} \right|_{x_j} = \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} + O(h^2).$$

Let $u_j$ denote the approximate value $u(x_j)$. Then truncating the $O(h^2)$ terms of the derivatives and plugging them into the differential equation, we find the following row equation at $x_j$ for $j = 1, \ldots, n-1$

$$-\left( \frac{a(x_{j+1}) - a(x_{j-1})}{4h^2} \right) (u_{j+1} - u_{j-1}) - \frac{a(x_j)}{h^2} (u_{j+1} - 2u_j + u_{j-1}) = f(x_j)$$

Since we know that $u(0) = u(1) = 0$, we get different equations for $j = 1$ and $j = n - 1$.

For $j = 1$, the equation is given by

$$-\left( \frac{a(x_2) - a(x_0)}{4h^2} \right) (u_2) - \frac{a(x_1)}{h^2} (u_2 - 2u_1) = f(x_1).$$

For $j = n - 1$, the equation is given by

$$-\left( \frac{a(x_n) - a(x_{n-2})}{4h^2} \right) (-u_{n-2}) - \frac{a(x_{n-1})}{h^2} (-2u_{n-1} + u_{n-2}) = f(x_{n-1}).$$

The resulting tridiagonal system has RHS $f(x_j)$ for $j = 1, 2, \cdots, n - 1$;

— diagonal entries:

$$2\frac{a(x_j)}{h^2}, \quad j = 1, \cdots, n - 1.$$
lower diagonal entries:

\[
\left( \frac{a(x_{j+1}) - a(x_{j-1})}{4h^2} \right) - \frac{a(x_j)}{h^2} = \frac{a(x_j)}{h^2} \cdot j = 2, \cdots, n - 1;
\]

upper diagonal entries:

\[
- \left( \frac{a(x_{j+1}) - a(x_{j-1})}{4h^2} \right) - \frac{a(x_j)}{h^2} = \frac{a(x_j)}{h^2} \cdot j = 1, \cdots, n - 2.
\]

(b) The Gershgorin theorem states that for any complex \( n \times n \) matrix \( A \), the eigenvalues lie within the collection of discs with radius \( R_i = \sum_{j \neq i} |a_{ij}| \) centered at \( a_{ii} \).

Applying this to the matrix, we find that the Gershgorin disc for the linear system are

\[
\left| \lambda - \frac{2a(x_1)}{h^2} \right| \leq \left| \frac{a(x_2) - a(0)}{4h^2} + \frac{a(x_1)}{h^2} \right|,
\]

\[
\left| \lambda - \frac{2a(x_j)}{h^2} \right| \leq 2 \left| \frac{a(x_{j+1}) - a(x_{j-1})}{4h^2} + \frac{a(x_j)}{h^2} \right|, \quad \text{for } j = 2, \ldots, n - 2,
\]

and

\[
\left| \lambda - \frac{2a(x_{n-1})}{h^2} \right| \leq \left| \frac{a(1) - a(x_{n-2})}{4h^2} + \frac{a(x_{n-1})}{h^2} \right|.
\]

Now we must find a disc which contains all of these. To do this, we utilize the fact that \( a(x) \geq \delta > 0 \). Let \( M = \max_{x \in [0,1]} |a(x)| \). Then

\[
\left| \lambda - \frac{2a(x_j)}{h^2} \right| \leq 2 \left| \frac{a(x_{j+1}) - a(x_{j-1})}{4h^2} + \frac{a(x_j)}{h^2} \right| \leq \frac{2}{h^2} \left( \frac{M - \delta}{4} + M \right).
\]

Now that we have an upper bound for the radius of all the disc, we note that \( \delta < a(x_j) \leq M \forall j \). Thus to make sure we capture all eigenvalues, we center the disc at \( \frac{\delta + M}{h^2} \) in the middle of \( \frac{2\delta}{h^2} \) and \( \frac{2M}{h^2} \), then add \( \frac{M - \delta}{h^2} \) to the radius we found above.

Finally the smallest disc that we find is

\[
\left| \lambda - \frac{M + \delta}{h^2} \right| \leq \frac{M - \delta}{h^2} + \frac{2}{h^2} \left( \frac{M - \delta}{4} + M \right) = \frac{1}{h^2} \left( 3.5M - 1.5\delta \right).
\]
**Problem 6: Numerical PDE**

Consider the equation

\[ u_t + au_x = 0, \quad a \in \mathbb{R}, \ t > 0, \]
\[ u(x, 0) = f(x). \tag{4} \]

The solution will be approximated using the Finite Difference Lax-Wendroff method

\[ v_j^{n+1} = v_j^n - \frac{a \Delta t}{2 \Delta x} (v_{j+1}^n - v_{j-1}^n) + a^2 \frac{(\Delta t)^2}{2(\Delta x)^2} (v_{j+1}^n - 2v_j^n + v_{j-1}^n) \]

where \( v_j^n = u(x_j, t_n) \) is a grid function, and \( \Delta x \) and \( \Delta t \) denote the spacing between grid points in the \( x \) and \( t \) directions.

**NOTE:** That there are two centered difference formulas used in the spatial direction.

(a) \[
\frac{v_j^{n+1} - v_j^n}{\Delta t} = u_t + \frac{a^2 \Delta t}{2} u_{xx} + \frac{a^3(\Delta t)^2}{6} u_{xxx} + O((\Delta t)^3)
\]

Rewrite this expression replacing the temporal derivatives \( u_{tt} \) and \( u_{ttt} \) in terms of spatial derivatives using equation (4).

(b) Determine the spatial and temporal orders of accuracy of the Lax-Wendroff method.

(c) Use von Neumann analysis to determine under what conditions the method is stable. (Hint: it is useful to look at the square of the amplification factor.)

Solution:

(a) \[
\frac{v_j^{n+1} - v_j^n}{\Delta t} = u_t + \frac{a^2 \Delta t}{2} u_{xx} + \frac{a^3(\Delta t)^2}{6} u_{xxx} + O((\Delta t)^3)
\]

(b) \[
\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} = u_x + \frac{(\Delta x)^2}{6} u_{xxx} + O((\Delta x)^4)
\]

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\]

(b) \[
\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} = u_x + \frac{(\Delta x)^2}{6} u_{xxx} + O((\Delta x)^4)
\]

\[
\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2} = u_{xx} + \frac{(\Delta x)^2}{12} u_{xxx} + O((\Delta x)^4)
\]

Plugging these two Taylor expansions and the solution to part (a) into the finite difference scheme, we get

\[
u_t + au_x = -\frac{a^3(\Delta t)^2}{6} u_{xxx} - \frac{a(\Delta x)^2}{6} u_{xxx} + O((\Delta t)^3) + O(\Delta x)^4
\]

So the method is \( O((\Delta x)^2 + (\Delta t)^2) \).

(c) We plug \( v_j^n = \xi n e^{ikx} \) into the finite difference method.

\[
v_j^{n+1} = v_j^n - \frac{a \Delta t}{2 \Delta x} (v_{j+1}^n - v_{j-1}^n) + a^2 \frac{(\Delta t)^2}{2(\Delta x)^2} (v_{j+1}^n - 2v_j^n + v_{j-1}^n)
\]

\[
\xi^{n+1} e^{ikx} = \xi^n e^{ikx} - \frac{a \Delta t}{2 \Delta x} (\xi^n e^{ik(x+\Delta x)} - \xi^n e^{ik(x-\Delta x)}) + a^2 \frac{(\Delta t)^2}{2(\Delta x)^2} (\xi^n e^{ik(x+\Delta x)} - 2\xi^n e^{ikx} + \xi^n e^{ik(x-\Delta x)})
\]

\[
\xi = 1 - \frac{a \Delta t}{2 \Delta x} \left( e^{ik\Delta x} - e^{-ik\Delta x} \right) + a^2 \frac{(\Delta t)^2}{2(\Delta x)^2} \left( e^{ik\Delta x} - 2 + e^{-ik\Delta x} \right)
\]

\[
= 1 - \frac{a \Delta t}{\Delta x} i \sin(k\Delta x) + a^2 \frac{(\Delta t)^2}{2(\Delta x)^2} (\cos(k\Delta x) - 1)
\]
Let $\kappa = \frac{a \Delta t}{\Delta x}$, then $\xi$ can be written in a slightly cleaner form.

$$\xi = 1 - \frac{a \Delta t}{\Delta x} i \sin(k \Delta x) + \kappa^2 (\cos(k \Delta x) - 1)$$

$$|\xi|^2 = 1 - 2\kappa^2(1 - \cos(k \Delta x)) + \kappa^2(1 - \cos^2(k \Delta x)) + \kappa^4(1 - \cos(k \Delta x))^2$$

$$= 1 - \kappa^2(1 - \kappa^2)(1 - \cos(k \Delta x))^2$$

$$= 1 - 4\kappa^2(1 - \kappa^2) \sin^4\left(\frac{1}{2}k \Delta x\right)$$

The method is stable when $|\xi|^2 \leq 1$ which matches the CFL condition requiring $|\kappa| \leq 1$. 