Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August 2020

Instructions. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your student ID number (not your name!) on your exam.

Problem 1: Root finding

Consider applying Newton's method to find a root of a real cubic polynomial $f(x)$.
(a) Give a heuristic (e.g. geometric) argument showing that if two roots coincide, there is precisely one starting guess x_0 (other than the double root) for which Newton will fail, and that this point separates the basins of attraction for the distinct roots.
(b) Suppose $x = \alpha$ is a root of multiplicity 2. Prove that if Newton's method converges to α, the convergence is first order.
(c) Propose a technique for recovering second order convergence of Newton's method for finding the double root α. Prove that your proposed method is in fact second order convergent.

Problem 2: Quadrature

Heun's method for the ODE $y'(t) = f(y)$ is

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(\tilde{y}_{n+1}) = f(y_n + hk_1) \\ y_{n+1} &= y_n + \frac{h}{2}\left[k_1 + k_2\right] \end{aligned}$$

where h is the step size. The ODE can also be written as an integral equation

$$y(t) = y(0) + \int_0^t f(y(s))\mathrm{d}s$$

(a) Explain how a single step of Heun's rule can be thought of as a left-endpoint quadrature followed by an approximate Trapezoid rule quadrature.

(b) Use the quadrature error formula for the simple Trapezoid rule, together with Taylor series, to derive a bound on the error in one step of Heun's method. You may assume that $y_n = y(t_n)$, i.e. there is no error in y_n.

Problem 3: Numerical Linear Algebra

The basic QR iteration for finding the eigenvalues of a real matrix A is

$$\mathrm{A}_{m-1} = \mathrm{Q}_{m-1}\mathrm{R}_{m-1}, \quad \mathrm{A}_m = \mathrm{R}_{m-1}\mathrm{Q}_{m-1}, \quad \mathrm{A}_0 = \mathrm{A}$$

where $\mathrm{A}_{m-1} = \mathrm{Q}_{m-1}\mathrm{R}_{m-1}$ is the QR factorization of A_{m-1}.

(a) Prove that the eigenvalues of A_m are the same as the eigenvalues of A.

(b) Explain how to construct an upper Hessenberg matrix H that has the same eigenvalues as A.

(c) Prove that if A is upper Hessenberg, then A_m is also upper Hessenberg. (You may assume that A is invertible, and that Q_m is upper Hessenberg whenever A_m is upper Hessenberg.)

Problem 4: Interpolation/Approximation

(a) Find the quadratic that interpolates the following temperature data: $T(-1) = 4$, $T(0) = 10$, $T(1) = 20$.

(b) Suppose that density is related to temperature via $\rho(T) = \rho_0 - \alpha T$, and that the total mass is known to be m

$$\int_{-1}^{1} \rho(T(x))\mathrm{d}x = m.$$

Find a cubic polynomial that interpolates the data from (a) and satisfies the above integral constraint, or explain why none exists. For part (b) let $\rho_0 = 1/2$, $m = 1$, and $\alpha = 1$.

(c) Consider the problem of both interpolating the data and satisfying the integral constraint for an arbitrary set of $n+1$ distinct interpolation nodes $x_0, \ldots, x_n$ using a polynomial of degree $\leq n+1$. When a solution exists, it must have the form

$$p(x) = q(x) + \sum_{j=0}^{n} T(x_j)\ell_j(x).$$

(i) When a solution does exist, give an explicit formula for $q(x)$.

(ii) Give an explicit criterion for when a solution does not exist.

Problem 5: Numerical ODE

Consider the boundary value problem

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = f(x), \quad u(0) = u(1) = 0$$

where $a(x) > \delta \geq 0$ is a bounded differentiable function in $[0, 1]$. We note that the above ODE can be written as

$$-\frac{da}{dx}\frac{du}{dx} - a(x)\frac{d^2u}{dx^2} = f(x), \quad u(0) = u(1) = 0.$$

We assume that, although $a(x)$ is available, an expression for its derivative, $\frac{da}{dx}$, is not available.

(a) Using finite differences and an equally spaced grid in $[0, 1]$, $x_l = hl$, $l = 0, \ldots, n$ and $h = 1/n$, we discretize the ODE to obtain a linear system of equations, yielding an $O(h^2)$ approximation of the ODE. After the application of the boundary conditions, the resulting coefficient matrix of the linear system is an $(n-1) \times (n-1)$ tridiagonal matrix.

Provide a derivation and write down the resulting linear system (by giving the expressions of the elements).

(b) Utilizing all the information provided, find a disc in $\mathbb{C}$, the smaller the better, that is guaranteed to contain all the eigenvalues of the linear system constructed in part (a).

Problem 6: Numerical PDE

Consider the equation

$$\begin{aligned} u_t + a u_x &= 0, \qquad a \in \mathbb{R},\ t > 0, \\ u(x,0) &= f(x). \end{aligned} \tag{1}$$

The solution will be approximated using the Finite Difference Lax-Wendroff method

$$v_j^{n+1} = v_j^n - \frac{a\Delta t}{2\Delta x}\left(v_{j+1}^n - v_{j-1}^n\right) + a^2\frac{(\Delta t)^2}{2(\Delta x)^2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right)$$

where $v_j^n = u(x_j, t_n)$ is a grid function, and Δx and Δt denote the spacing between grid points in the x and t directions, respectively.

NOTE: That there are two centered difference formulas used in the spatial direction.

(a) Consider Taylor expansion with respect to time

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = u_t + \frac{\Delta t}{2}u_{tt} + \frac{(\Delta t)^2}{6}u_{ttt} + O((\Delta t)^3)$$

Rewrite this expression replacing the temporal derivatives u_{tt} and u_{ttt} in terms of spatial derivatives using equation (1).

(b) Determine the spatial and temporal orders of accuracy of the Lax-Wendroff method.

(c) Use von Neumann analysis to determine under what conditions the method is stable. How does this relate to the CFL condition?