Instructions. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your student ID number (not your name!) on your exam.

Problem 1: Root finding
Consider applying Newton’s method to find a root of a real cubic polynomial \( f(x) \).
(a) Give a heuristic (e.g. geometric) argument showing that if two roots coincide, there is precisely one starting guess \( x_0 \) (other than the double root) for which Newton will fail, and that this point separates the basins of attraction for the distinct roots.
(b) Suppose \( x = \alpha \) is a root of multiplicity 2. Prove that if Newton’s method converges to \( \alpha \), the convergence is first order.
(c) Propose a technique for recovering second order convergence of Newton’s method for finding the double root \( \alpha \). Prove that your proposed method is in fact second order convergent.

Problem 2: Quadrature
Heun’s method for the ODE \( y'(t) = f(y) \) is
\[
\begin{align*}
k_1 &= f(y_n) \\
k_2 &= f(\tilde{y}_{n+1}) = f(y_n + hk_1) \\
y_{n+1} &= y_n + \frac{h}{2} [k_1 + k_2]
\end{align*}
\]
where \( h \) is the step size. The ODE can also be written as an integral equation
\[
y(t) = y(0) + \int_0^t f(y(s)) ds
\]
(a) Explain how a single step of Heun’s rule can be thought of as a left-endpoint quadrature followed by an approximate Trapezoid rule quadrature.
(b) Use the quadrature error formula for the simple Trapezoid rule, together with Taylor series, to derive a bound on the error in one step of Heun’s method. You may assume that \( y_n = y(t_n) \), i.e. there is no error in \( y_n \).

Problem 3: Numerical Linear Algebra
The basic QR iteration for finding the eigenvalues of a real matrix \( A \) is
\[
A_{m-1} = Q_{m-1}R_{m-1}, \quad A_m = R_{m-1}Q_{m-1}, \quad A_0 = A
\]
where \( A_{m-1} = Q_{m-1}R_{m-1} \) is the QR factorization of \( A_{m-1} \).
(a) Prove that the eigenvalues of \( A_m \) are the same as the eigenvalues of \( A \).
(b) Explain how to construct an upper Hessenberg matrix $H$ that has the same eigenvalues as $A$.

(c) Prove that if $A$ is upper Hessenberg, then $A^m$ is also upper Hessenberg. (You may assume that $A$ is invertible, and that $Q_m$ is upper Hessenberg whenever $A^m$ is upper Hessenberg.)

**Problem 4: Interpolation/Approximation**

(a) Find the quadratic that interpolates the following temperature data: $T(-1) = 4$, $T(0) = 10$, $T(1) = 20$.

(b) Suppose that density is related to temperature via $\rho(T) = \rho_0 - \alpha T$, and that the total mass is known to be $m$

$$\int_{-1}^{1} \rho(T(x))dx = m.$$ 

Find a cubic polynomial that interpolates the data from (a) and satisfies the above integral constraint, or explain why none exists. For part (b) let $\rho_0 = 1/2$, $m = 1$, and $\alpha = 1$.

(c) Consider the problem of both interpolating the data and satisfying the integral constraint for an arbitrary set of $n + 1$ distinct interpolation nodes $x_0, \ldots, x_n$ using a polynomial of degree $\leq n + 1$. When a solution exists, it must have the form

$$p(x) = q(x) + \sum_{j=0}^{n} T(x_j)\ell_j(x).$$

(i) When a solution does exist, give an explicit formula for $q(x)$.

(ii) Give an explicit criterion for when a solution does not exist.

**Problem 5: Numerical ODE**

Consider the boundary value problem

$$-\frac{d}{dx} \left( a(x) \frac{du}{dx} \right) = f(x), \quad u(0) = u(1) = 0$$

where $a(x) > \delta \geq 0$ is a bounded differentiable function in $[0, 1]$. We note that the above ODE can be written as

$$\frac{da}{dx} \frac{du}{dx} - a(x) \frac{d^2u}{dx^2} = f(x), \quad u(0) = u(1) = 0.$$ 

We assume that, although $a(x)$ is available, an expression for its derivative, $\frac{da}{dx}$, is not available.

(a) Using finite differences and an equally spaced grid in $[0, 1]$, $x_l = hl$, $l = 0, \ldots, n$ and $h = 1/n$, we discretize the ODE to obtain a linear system of equations, yielding an $O(h^2)$ approximation of the ODE. After the application of the boundary conditions, the resulting coefficient matrix of the linear system is an $(n-1) \times (n-1)$ tridiagonal matrix.

Provide a derivation and write down the resulting linear system (by giving the expressions of the elements).

(b) Utilizing all the information provided, find a disc in $\mathbb{C}$, the smaller the better, that is guaranteed to contain all the eigenvalues of the linear system constructed in part (a).
Problem 6: Numerical PDE

Consider the equation

\[ u_t + au_x = 0, \quad a \in \mathbb{R}, \; t > 0, \]
\[ u(x, 0) = f(x). \quad (1) \]

The solution will be approximated using the Finite Difference Lax-Wendroff method

\[ v^{n+1}_j = v^n_j - \frac{a\Delta t}{2\Delta x} (v^n_{j+1} - v^n_{j-1}) + a^2 \frac{(\Delta t)^2}{2(\Delta x)^2} (v^n_{j+1} - 2v^n_{j} + v^n_{j-1}) \]

where \( v^n_j = u(x_j, t_n) \) is a grid function, and \( \Delta x \) and \( \Delta t \) denote the spacing between grid points in the \( x \) and \( t \) directions, respectively.

**NOTE:** That there are two centered difference formulas used in the spatial direction.

(a) Consider Taylor expansion with respect to time

\[ \frac{v^{n+1}_j - v^n_j}{\Delta t} = u_t + \frac{\Delta t}{2} u_{tt} + \frac{(\Delta t)^2}{6} u_{ttt} + O((\Delta t)^3) \]

Rewrite this expression replacing the temporal derivatives \( u_{tt} \) and \( u_{ttt} \) in terms of spatial derivatives using equation (1).

(b) Determine the spatial and temporal orders of accuracy of the Lax-Wendroff method.

(c) Use von Neumann analysis to determine under what conditions the method is stable. How does this relate to the CFL condition?