Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
January, 2019

Instructions. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your student ID number (not your name!) on your exam.

1. **Root Finding.** Consider the fixed point iteration scheme

$$x_{n+1} = g(x_n).$$

   (a) State the necessary conditions for the convergence of such a scheme to fixed point $x = \alpha$.

   (b) Given

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$

   Find an upper bound for the absolute error $|\alpha - x_n|$.

   (c) Derive from first principles the expression that shows the rootfindng method to be pth order convergent.

   (d) Consider the following iteration for calculating $\gamma^{1/3}$:

$$x_{n+1} = a x_n + b\frac{\gamma}{x_n^2} + c\frac{\gamma^2}{x_n^5}$$

   Assuming that this iterative scheme converges for x_0 sufficiently close to $\gamma^{1/3}$ determine a, b, c such that the method has the highest possible convergence rate.

2. **Linear Alegbra.** Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite (spd), and consider the following iteration.

> Choose $A_0 = A$
> **for** $k = 0, 1, 2, \ldots$
>   Compute the Cholesky factor L_k of A_k (so $A_k = L_k L_k^T$)
>   Set $A_{k+1} = L_k^T L_k$
> **end**

Here L_k is lower triangular with positive diagonal elements.

(a) Show that A_k is similar to A, and that A_k is spd (the iteration is therefore well-defined).

(b) Now consider the special case of a 2×2 spd matrix,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad a \geq c,$$

For this matrix, perform one step of the algorithm and write down A_1.

(c) Use the result from **(b)** to argue that A_k converges to $\mathrm{diag}(\lambda_1, \lambda_2)$, where the eigenvalues of A are ordered as $\lambda_1 \geq \lambda_2 > 0$.

3. **Numerical quadrature.**

(a) State the simple (one panel) midpoint and trapezoidal rules for approximating the integral $\int_0^h f(x)\,dx$.

(b) Derive the error formulas for both methods.

(c) Let $f(x) = x^3$ and show that it is possible to combine the results from the two methods so that the answer is exact.

(d) Show that you also get the exact answer if you perform Richardson extrapolation (using the error expansion from the composite rule) using the answers obtained by the trapezoidal method with one panel (as above) and the trapezoidal method with two panels of equal size $h/2$.

4. **Interpolation/Approximation.** We are given three data points as follows:

x	-1	0	1
y	1	-1	-1

Determine the interpolating polynomial of lowest degree possible using

(a) Lagrange's interpolation formula,

(b) Newton's interpolation formula,

(c) Stirling's interpolation formula,

Verify that the answers you get agree.

(d) Quote the formula for the error, applied to this special case.

Hint: In standard operator notation on a grid with spacing h, Stirling's interpolation formula can be written

$$f(x_0 + th) = f_0 + t\mu\delta f_0 + \frac{t^2}{2!}\delta^2 f_0 + \frac{t(t^2-1)}{3!}\mu\delta^3 f_0 + \frac{t^2(t^2-1)}{4!}\delta^4 f_0 + \cdots$$

5. **ODEs** Consider the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$. The Milne method is a linear multistep method defined by

$$y_n = y_{n-2} + \int_{t_{n-2}}^{t_n} P(t)dt$$

where $P(t)$ is the unique quadratic polynomial that interpolates f at the points t_{n-2}, t_{n-1}, t_n, and $t_n = hn$.

(a) Derive the formula for this method.

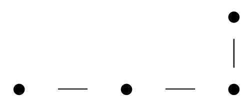
(b) Find the leading order term in the local truncation error of this method. What is the order of this method?

(c) Is this method 0-stable, strongly stable? Why?

6. **PDEs** Consider the following finite difference scheme:

$$\frac{u(x,t+k)-u(x,t)}{k}+\frac{\frac{3}{2}u(x,t)-2u(x-h,t)+\frac{1}{2}u(x-2h,t))}{h}=0$$

Graphically, we illustrate its stencil as shown below



(a) Determine which PDE the scheme is consistent with,

(b) Determine its order of accuracy in time and in space (when applied to the the PDE from (a)),

(c) Use von Neumann analysis to determine the scheme's stability.

Hint to part (c): The figure below shows the curve traced out by $f(s)=\frac{3}{2}-2e^{is}+\frac{1}{2}e^{-2is}$ for $-\pi \leq s \leq \pi$.