Department of Applied Mathematics

Preliminary Examination in Numerical Analysis

January 11, 2021, 9 am – 12 noon.

Submit solutions to four (and no more) of the following six problems. Show all your work, and justify all your answers. Start each problem on a new page, and write on one side only. No calculators allowed.
Do not write your name on your exam. Instead, write your student number on each page.

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1. Root finding

Consider Newton's method for solving the equation $\sin x = 0$ in the interval $(-\pi/2, \pi/2)$ starting with the initial approximation x_0 such that $\tan x_0 = 2x_0$ (nb. $x_0 \approx \pm 1.1656$).

a. What is the result of this iteration?

b. What is the result of the iteration if the initial approximation $\tilde{x}_0$ satisfies $|\tilde{x}_0| < |x_0|$?

c. What is the result of the iteration if the initial approximation $\tilde{x}_0$ satisfies $|\tilde{x}_0| > |x_0|$?

2. Quadrature

a. What is the largest step size that makes the trapezoidal rule exact for trigonometric polynomials of the form

$$\sum_{n=-N}^{N} c_n e^{int}, \quad t \in [0, 2\pi).$$

b. Show that the formula

$$\int_{-1}^{1} f(x)(1-x^2)^{-1/2}\, dx = \frac{\pi}{N} \sum_{n=1}^{N} f\left(\cos\left(\frac{2n-1}{2N}\pi\right)\right)$$

is exact for all polynomials *f* of degree $2N-1$.

3. Linear Algebra

a. Define what is meant by a matric being *Hermitian*, and show that such a matrix has only real eigenvalues.

b. A matrix *A* is called *circulant* if its elements $a_{i,j}$ are all the same whenever $(i-j) \bmod N$ is the same. In other words, each row is the same as the row above shifted periodically one step to the right. Show that such a matrix can be diagonalized by similarity transforming it using the DFT (Discrete Fourier Transform) matrix, as given by

$$U = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{bmatrix},$$

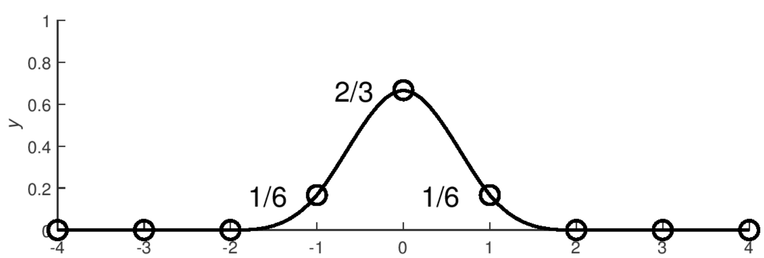
where ω is the N^{th} root of unity.

c. The matrix $A = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$ is both Hermitian and circulant.

Determine all its eigenvalues.

4. <u>**Interpolation / Approximation**</u>

The figure below illustrates a cubic *B*-spline on a unit-spaced grid, uniquely defined when using its standard normalization.



a. Tell what the defining property is of a *B-spline* (as opposed to any other spline). Also tell what is the customary normalization of any *B*-spline.

b. Verify that the *B*-spline in the figure can be written explicitly as

$$B(x) = \frac{1}{12}\left(1\cdot|x+2|^3 - 4\cdot|x+1|^3 + 6\cdot|x|^3 - 4\cdot|x-1|^3 + 1\cdot|x-2|^3\right). \quad (1)$$

c. Translates of *B*-splines form an excellent set of basis functions for representing a general spline. Consider the nodes $x_i = i, i = 0,1,2,\ldots,N$ with matching function values y_i, and let $B_i(x)$ denote the *B*-spline centered at $x = i,\ \ i = -1,0,1,2,\ldots,N+1.$ Write out the linear system that needs to be solved for obtaining the *B*-spline coefficients for the *natural* cubic spline that obeys this data.

<u>Hint:</u> For the *B*-spline (as given in (1)), $B''(x)$ takes the values {1,-2, 1} at $x = \{-1, 0, 1\}$.

5. **Numerical ODEs**

a. Define what is meant by the stability domain of an ODE solver.

b. Determine the stability domains for the Forward Euler (FE) and Backward Euler (BE) methods (first order Adams-Bashforth and Adams-Moulton methods, respectively).

c. Suppose one uses FE as a predictor and BE as a corrector. What is the order of accuracy of the resulting method? Either derive it, or quote a specific, more general theorem.

d. Give an equation for the stability domain of the FE – BE predictor corrector method. Determine what (if any) intervals along the real and imaginary axes fall within this domain.

6. **Numerical PDEs**

Consider the Crank-Nicolson method

$$u_j^{(n+1)} - u_j^{(n)} = \frac{1}{2}\frac{k}{h^2}\left(\left(u_{j-1}^{(n+1)} - 2u_j^{(n+1)} + u_{j+1}^{(n+1)}\right) + \left(u_{j-1}^{(n)} - 2u_j^{(n)} + u_{j+1}^{(n)}\right)\right)$$

for the heat equation

$$\begin{cases} u_t = u_{xx},\, t \geq 0,\, x \in [0, 2\pi] \\ u(x,0) = u_0(x) \\ u(x+2\pi, t) = u(x,t),\, t \geq 0 \end{cases} .$$

a. Show that the scheme is unconditionally stable.

b. Show that it is second order accurate in both space and time.