Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August 2023

Instructions. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. All problems have equal value.

Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted.
Write your student ID number (not your name!) on your exam.

Problem 1: Root finding

(a) Consider the following iteration schemes of the form $x_{n+1} = f(x_n)$ each with a proposed fixed point α. Which of the following will converge (provided x_0 is sufficiently close to α)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence

(i) $x_{n+1} = -16 + 6x_n + \dfrac{12}{x_n}, \qquad \alpha = 2$

(ii) $x_{n+1} = \dfrac{2}{3}x_n + \dfrac{1}{x_n^2}, \qquad \alpha = 3^{1/3}$

(iii) $x_{n+1} = \dfrac{12}{1 + x_n}, \qquad \alpha = 3$

(b) Consider an analytic function $f(x)$ such that the fixed point iteration

$$x_{n+1} = f(x_n)$$

for any initial value of $x_0 \neq 0$ eventually hops between $+1$ and -1. Describe the properties of $f(x)$ for $|x| = 1$, $|x| < 1$, and $|x| > 1$ that would make this limiting sequence possible. Sketch such a function.

Problem 2: Interpolation/Approximation

(a) Define what it means for a polynomial $p_N(x)$ to be a minimax approximation of degree N.

(b) Find the first degree Taylor polynomial approximating e^x in the interval $[-1, 1]$ centered at $a = 0$. Then find the maximum norm of the error in this approximation.

(c) Find the first degree polynomial least squares approximation of the function e^x that minimizes the error in the following norm:

$$\|f - g\| := \sqrt{\int_{-1}^{1} |f(x) - g(x)|^2 dx}$$

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(d) Create the polynomial that interpolates e^x with the nodes $x_0 = -1$ and $x_1 = 1$.

(e) Which of the three polynomials that you created is the closest to the optimal approximating polynomial in the minimax sense and why?

Problem 3: Quadrature

Gaussian quadratures that approximate as follows

$$\int_{-1}^{1} f(x)dx \sim \sum_{k=0}^{N} w_k f(x_k)$$

where the quadrature nodes include the endpoints (i.e. $x_0 = -1$ and $x_N = 1$) are called *Gauss-Legendre-Lobatto* quadratures.

(a) Show that if the interior nodes $x_1, \ldots, x_{N-1}$ in the quadrature are given by the roots of $P'_N(x)$ where $P_N(x)$ is the N^{th} degree Legendre polynomial, then the quadrature is exact for polynomials up to degree $2N-1$.
Hint: The following recurrence relation is true:

$$(x^2-1)P'_N(x) = xP_N(x) - P_{N-1}(x)$$

(b) Find the $4-$point Gauss-Legendre-Lobatto quadrature (nodes and weights) for approximating the integral $\int_{-1}^{1} f(x)dx$.
It is enough to set up a closed formula which evaluates each of the weights independently.
Hint: The three term recursion for Legendre polynomials is given by

$$P_0(x) = 1,\ P_1(x) = x,\ kP_k(x) - (2k-1)xP_{k-1}(x) + (k-1)P_{k-2}(x) = 0$$

Problem 4: Linear algebra

In some computational settings a block LU decomposition is useful. In this problem you will build the block LU factorization of a matrix and determine the computational complexity of using such a technique for solving a linear system.

(a) Consider the $2n \times 2n$ block matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where each block is an $n \times n$ matrix. Derive the matrices $\hat{\mathbf{L}}_{21}$ and $\hat{\mathbf{A}}_{22}$ such that

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{L}}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \hat{\mathbf{A}}_{22} \end{bmatrix}.$$

(b) What is the computational cost in the big O sense for constructing the factorization and solving a linear system $\mathbf{Ax} = \mathbf{b}$ with the precomputed factorization? The answer should be in terms of the number of blocks and the size of the blocks. Provide justification for your answer.
Note: you should assume that any inverse created while making the factorization is available for the solve stage.

(c) Building off your work in part (a), derive the formula for a $3n \times 3n$ block LU factorization of a matrix $\mathbf{A}$ where each of blocks is of size $n \times n$. This means that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix}$$

where each bock is an $n \times n$ matrix. *Hint: Blocking will be helpful.*

(d) What is the computational cost in the big O sense for constructing the factorization and solving a linear system $\mathbf{Ax} = \mathbf{b}$ with the precomputed factorization in part (b)? The answer should be in terms of the number of blocks and the size of the blocks. Provide justification for your answer.
Note: you should assume that any inverse created while making the factorization is available for the solve stage.

Problem 5: Numerical ODEs

Consider the following implicit three-step method

$$y_{n+3} - y_n = h\left[\mu f(t_{n+3}, y_{n+3}) + \frac{9}{8} f(t_{n+2}, y_{n+2}) + \frac{9}{8} f(t_{n+1}, y_{n+1}) + \frac{3}{8} f(t_n, y_n)\right]$$

with undetermined coefficient μ to be designed to numerically solve

$$y' = f(t, y), \quad y(t_0) = y_0$$

i) Determine the value of μ that makes this scheme consistent.

ii) Determine the order of the consistent scheme by looking at the truncation error.

iii) Is the consistent scheme convergent?

Problem 6: Numerical PDEs

Consider the initial value problem for one-dimensional wave propagation

$$\partial_{tt} u = c^2 \partial_{xx} u, \quad t \geq 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x).$$

(a) An explicit time-stepping numerical method using central differences to discretize space and time derivatives gives

$$U(x, t + k) - 2U(x, t) + U(x, t - k) = \alpha^2 \left[U(x + h, t) - 2U(x, t) + U(x - h, t)\right]$$

where $\alpha = c(h/k)$, $k = \Delta t$ and $h = \Delta x$. Assuming $U(x, t) = \zeta^{t/k} e^{i\omega h}$, a Von Newmann analysis performed on this disctetization gives the amplification equations

$$\zeta^2 - 2\beta\zeta + 1 = 0 \quad \text{where} \quad \beta = 1 - 2\alpha^2 \sin^2\left(\frac{\omega h}{2}\right).$$

Show that the scheme is conditionally stable and establish explicitly the stability condition.

(b) Consider the following implicit time-stepping numerical method using central differences to discretize space and time derivatives

$$
\begin{aligned}
U(x,t+k) - 2U(x,t) + U(x,t-k) &= \frac{\alpha^2}{2}\left[U(x+h,t-k) - 2U(x,t-k) + U(x-h,t-k)\right] && (1)\\
&+\frac{\alpha^2}{2}\left[U(x+h,t+k) - 2U(x,t+k) + U(x-h,t+k)\right] && (2)
\end{aligned}
$$

(i) **Find** the amplification equation. ($\beta = 1 + 2\alpha^2 \sin^2\left(\frac{\omega h}{2}\right)$ is a useful definition).

(ii) **Determine** whether the scheme is conditionally or absolutely stable.