Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August 2021

Instructions. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. All problems have equal value.

Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted.
Write your student ID number (not your name!) on your exam.

Problem 1: Root finding
Consider the fixed point iteration scheme

$$x_{n+1} = g(x_n).$$

a) State the necessary conditions for the convergence of such a scheme to fixed point $x = \alpha$.

b) Find an upper bound for the absolute error $|\alpha - x_n|$.

c) Derive from first principles the expression that shows the method to be pth order convergent.

d) Consider the following iteration for calculating $\gamma^{1/3}$:

$$x_{n+1} = ax_n + b\frac{\gamma}{x_n^2} + c\frac{\gamma^2}{x_n^5}$$

Assuming that this iterative scheme converges for x_0 sufficiently close to $\gamma^{1/3}$ determine a, b, c such that the method has the highest possible convergence rate.

Problem 2: Quadrature
Consider $\{p_i(x)\}_{i=0}^{\infty}$, a family of orthogonal polynomials associated with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, w(x)\, dx, \quad w(x) > 0 \text{ for } x \in (-1, 1),$$

where $p_i(x)$ is a polynomial of the degree i. Let x_0, x_1,…,x_n be the roots of $p_{n+1}(x)$. Construct an orthonormal basis in the subspace of polynomials of degree less or equal n such that, for any polynomial in this subspace, the coefficients of its expansion into the basis are equal to the scaled values of this polynomial at the nodes x_0, x_1,…,x_n.

Problem 3: Numerical Linear Algebra
An algorithm (attributed to Eudoxos) generating a sequence of increasingly accurate approximations to the length of the diagonal of the unit square has been know in ancient Greece. Starting with $p_0 = q_0 = 1$, and defining the iteration as $p_{n+1} = p_n + q_n$, and $q_{n+1} = p_{n+1} + p_n$, $n = 0, 1, 2, \ldots$, then the ratio q_n/p_n converses to $\sqrt{2}$ as $n \to \infty$.

(a) Prove convergence of this algorithm using the power method.

(b) Give an expression that determines how many iterations are needed in order to achieve the accuracy

$$\left|\sqrt{2} - \frac{q_n}{p_n}\right| \leq 10^{-8}?$$

Problem 4: Interpolation/Approximation

(a) Let $f : [a, b] \to \mathbb{R}$ be a smooth function. Given the points $a < x_0 < x_1 < \cdots < x_n < b$, show there is a unique polynomial that interpolates the data $(x_i, f(x_i)), i = 0, \ldots, n$.

(b) Let $\epsilon > 0$ and consider the three data values $(0, f(0)), (\epsilon, f(\epsilon))$ and $(1, f(1))$. Let q be the polynomial that arises as the limit of the second order polynomial interpolant of the data as $\epsilon \to 0$.

What is the degree of q?

What data (if any) does q interpolate?

What data (if any) does q' interpolate?

(c) Given the the points $a < x_0 < x_1 < \cdots < x_n < b$, denote by Ψ the function

$$\Psi(x) = \sum_{j=0}^{n} c_j e^{jx}$$

such that

$$\Psi(x_i) = f(x_i), \quad i = 0, \ldots, n$$

Is the choice of interpolations constants $c_0, \ldots, c_n$ unique? Provide justification for your answer.

Problem 5: Numerical ODE
Consider the following method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right], \tag{1}$$

for $n = 1, 2, 3, \ldots$, which approximate the solution of ordinary differential equations of the form

$$y' = f(x, y).$$

(a) Determine the order of the method in equation (1).

(b) State the conditions required for the method to be convergent.

(c) Determine if the method is convergent.

(d) Determine the region of absolute stability for (1).

(e) State what is required for a method to be A-stable.

(f) Is this method in equation (1) A-stable?

Problem 6: Numerical PDE

Given the one-dimensional diffusion equation

$$\partial_t u = \partial_{xx} u$$

consider the following fully discretized scheme

$$\begin{aligned} U(x,t+k) - \frac{1}{2}(\mu - \xi)\left[U(x-h,t+k) - 2U(x,t+k) + U(x+h,t+k)\right] = \\ U(\ell,n) + \frac{1}{2}(\mu + \xi)\left[U(x-h,t) - 2U(x,t) + U(x+h,t)\right] \end{aligned} \tag{2}$$

$\mu = k/h^2$ and $\xi =$ constant.

(a) Use Von Neumann analysis to obtain the stability condition that relates the allowable time step k and space step h.

(b) Show that a Taylor expansion of (2) results in

$$\begin{aligned} k\partial_t u + \sum_{j=2}^{\infty} \frac{k^j}{j!} \partial_t^j u - \frac{1}{2}(\mu - \xi)\left[\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}(1 + (-1)^i)\frac{h^i}{i!}\frac{k^j}{j!}\partial_x^i \partial_t^j u\right] = \\ \mu h^2 \partial_x^2 u + \mu\left[\sum_{i=4}^{\infty}(1 + (-1)^i)\frac{h^i}{i!}\partial_x^i u\right] \end{aligned}$$