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An elementary proof that almost all real numbers are normal

Ferdinánd Filip
Department of Mathematics,
Faculty of Education,
J. Selye University,
Rolníckej školy 1519,
945 01, Komárno, Slovakia
email: filip.ferdinand@selyeuni.sk

Jan Šustek
Department of Mathematics and
Institute for Research and
Applications of Fuzzy Modeling,
Faculty of Science,
University of Ostrava,
30. dubna 22,
701 03 Ostrava 1, Czech Republic
email: jan.sustek@osu.cz

Abstract. A real number is called normal if every block of digits in its expansion occurs with the same frequency. A famous result of Borel is that almost every number is normal. Our paper presents an elementary proof of that fact using properties of a special class of functions.

1 Introduction

The concept of normal number was introduced by Borel. A number is called normal if in its base b expansion every block of digits occurs with the same frequency. More exact definition is

Definition 1 *A real number $x \in (0,1)$ is called simply normal to base $b \geq 2$ if its base b expansion is $0.c_1c_2c_3\dots$ and*

$$\lim_{N\to\infty} \frac{\#\{n \leq N \mid c_n = a\}}{N} = \frac{1}{b} \quad \textit{for every } a \in \{0,\dots,b-1\}.$$

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A number is called normal to base b *if for every block of digits* $a_1 \dots a_L$*,* $L \geq 1$

$$\lim_{N\to\infty} \frac{\#\{n \leq N - L \mid c_{n+1} = a_1, \dots, c_{n+L} = a_L\}}{N} = \frac{1}{b^L}\,.$$

A number is called absolutely normal *if it is normal to every base* $b \geq 2$*.*

A famous result of Borel [1] is

Theorem 1 *Almost every real number is absolutely normal.*

This theorem can be proved in many ways. Some proofs use uniform distribution [5], combinatorics [7], probability [8] or ergodic theory [2]. There are also some elementary proofs almost avoiding higher mathematics. Kac [3] proves the theorem for simply normal numbers to base 2 using Rademacher functions and Beppo Levi's Theorem. Nillsen [6] also considers binary case. He uses series of integrals of step functions and avoids usage of measure theory in the proof by defining a null set in a different way. Khoshnevisan [4] makes a survey about known results on normal numbers and their consequences in diverse areas in mathematics and computer science.

This paper presents another elementary proof of Theorem 1. Our proof is based on the fact that a bounded monotone function has finite derivative in almost all points. We also use the fact that a countable union of null sets is a null set.

Here is a sketch of the proof. We introduce a special class of functions. In Section 2 we prove elementary properties of the functions $\mathcal{F}$. We prove boundedness and monotonicity and assuming that the derivative $\mathcal{F}'(x)$ exists in point x we prove that the product (5) has finite value. We deduce that the product (5) has finite value for almost every x. In Section 3 we prove that every non-normal number belongs to some set P. We take a particular function $\mathcal{F}$. We finish the proof by showing that for elements of P the product (5) does not have finite value.

For the proof of Theorem 1 it is obviously sufficient to consider only numbers in the interval $(0, 1)$.

Definition 2 *Let* $\mathbf{b} = \{b_k\}_{k=1}^{\infty}$ *be a sequence of integers* $b_k \geq 2$*. Let* $\boldsymbol{\omega} = \{\omega_k\}_{k=1}^{\infty}$ *be a sequence of divisions of the interval* $[0, 1]$*,*

$$\omega_k = \{f_k(c)\}_{c=0}^{b_k}\,, \quad f_k(0) = 0\,, \quad f_k(c) < f_k(c+1)\,, \quad f_k(b_k) = 1\,.$$

Put

$$\Delta_k(c) := f_k(c+1) - f_k(c)\,.$$

Function $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\colon [0,1] \to [0,\infty)$ *corresponding to* $\mathbf{b}$ *and* $\boldsymbol{\omega}$ *is defined as follows. For* $x \in [0,1)$*, let*

$$x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k} \tag{1}$$

be its $\{b_k\}_{k=1}^{\infty}$*-Cantor series. Then*

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) := \sum_{n=1}^{\infty} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k)\,.$$

We define $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(1) = 1$.

The reason for defining $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(1) = 1$ is that the actual range of $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$ is $\subseteq [0,1]$. This is proved in Lemma 2.

2 Properties of the function $\mathcal{F}$

In this section we derive some basic properties of a general function $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$.

Lemma 1 allows us to express a particular value $\mathcal{F}(x)$ in terms of values of some other function $\mathcal{F}$.

For $N \in \mathbb{N}$ define $\mathbf{b}^{(N)} := \{b_n^{(N)}\}_{n=1}^{\infty}$, $\boldsymbol{\omega}^{(N)} := \{\omega_n^{(N)}\}_{n=1}^{\infty}$ and $\{\Delta_n^{(N)}\}_{n=1}^{\infty}$ by

$$b_n^{(N)} := b_{N+n}\,, \quad \omega_n^{(N)} := \omega_{N+n}\,, \quad \Delta_n^{(N)} := \Delta_{N+n}\,.$$

Moreover, for $x = \sum_{n=1}^{\infty} \frac{c_k}{\prod_{k=1}^{n} b_k} \in (0,1)$ define

$$x^{(N)} := \sum_{n=1}^{\infty} \frac{c_{N+n}}{\prod_{k=1}^{n} b_k^{(N)}}\,.$$

Lemma 1 (Shift property) *We have*

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) = \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + \prod_{k=1}^{N} \Delta_k(c_k) \cdot \mathcal{F}_{\mathbf{b}^{(N)},\boldsymbol{\omega}^{(N)}}(x^{(N)})\,.$$

Proof. An easy computation yields

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) = \sum_{n=1}^{\infty} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k)$$

$$= \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k)$$
$$+ \prod_{k=1}^{N} \Delta_k(c_k) \sum_{n=1}^{\infty} f_{N+n}(c_{N+n}) \prod_{k=1}^{n-1} \Delta_{N+k}(c_{N+k})$$
$$= \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + \prod_{k=1}^{N} \Delta_k(c_k) \cdot \mathcal{F}_{\mathbf{b}^{(N)},\boldsymbol{\omega}^{(N)}}(x^{(N)}) .$$

■

Lemma 2 (Range) *For* $x \in [0,1]$ *the value* $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) \in [0,1]$.

Proof. First we prove that for every $\mathbf{b}, \boldsymbol{\omega}$ and every $x = \sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}$

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\Big(\sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}\Big) = \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) \le 1 . \qquad (2)$$

We will proceed by induction on N.

For $N = 1$ we have $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}\big(\frac{c_1}{b_1}\big) = f_1(c_1) \le 1$.

For $N+1$ we use Lemma 1. By the induction assumption we have

$$\mathcal{F}_{\mathbf{b}^{(1)},\boldsymbol{\omega}^{(1)}}(x^{(1)}) \le 1 .$$

Hence

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) = f_1(c_1) + \Delta_1(c_1)\mathcal{F}_{\mathbf{b}^{(1)},\boldsymbol{\omega}^{(1)}}(x^{(1)}) \le f_1(c_1) + \Delta_1(c_1) = f_1(c_1+1) \le 1 .$$

Now we use (2) and pass to the limit $N \to \infty$. For $x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k}$ we have

$$\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}(x) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) \le 1 .$$

■

Lemma 3 (Monotonicity) *The function* $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$ *is nondecreasing.*

Proof. Let $0 \le x < y < 1$ be two numbers with

$$x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k} \quad \text{and} \quad y = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{k=1}^{n} b_k}.$$

We prove that $\mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(x) \le \mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(y)$.

Let N be the integer such that $c_n = d_n$ for $n \le N-1$ and $c_N < d_N$. Then Lemmas 1 and 2 imply

$$\begin{aligned}
\mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(x) &= \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + \prod_{k=1}^{N} \Delta_k(c_k) \cdot \mathcal{F}_{\mathfrak{b}^{(N)},\boldsymbol{\omega}^{(N)}}(x^{(N)}) \\
&\le \sum_{n=1}^{N} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + \prod_{k=1}^{N} \Delta_k(c_k) \\
&= \sum_{n=1}^{N-1} f_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k) + f_N(c_N+1) \prod_{k=1}^{N-1} \Delta_k(c_k) \\
&\le \sum_{n=1}^{N-1} f_n(d_n) \prod_{k=1}^{n-1} \Delta_k(d_k) + f_N(d_N) \prod_{k=1}^{N-1} \Delta_k(d_k) \\
&\le \sum_{n=1}^{\infty} f_n(d_n) \prod_{k=1}^{n-1} \Delta_k(d_k) = \mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(y). \qquad \blacksquare
\end{aligned}$$

For $k \in \mathbb{N}$ and $c \in \{0, \dots, b_k\}$ define $\overline{f}_k(c) := 1 - f_k(b_k - c)$. Put $\overline{\boldsymbol{\omega}} := \{\{\overline{f}_k(c)\}_{c=0}^{b_k}\}_{k=1}^{\infty}$.

Lemma 4 (Symmetry) *For every* $x = \sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}$ *we have*

$$\mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(1-x) = 1 - \mathcal{F}_{\mathfrak{b},\overline{\boldsymbol{\omega}}}(x). \tag{3}$$

Proof. We have

$$\overline{\Delta}_k(c) = \overline{f}_k(c+1) - \overline{f}_k(c) = \Delta_k(b_k - c - 1).$$

Now we will proceed by induction.

For $N = 1$ we have

$$\mathcal{F}_{\mathfrak{b},\boldsymbol{\omega}}(1-x) = f_1(b_1 - c_1) = 1 - \overline{f}_1(c_1) = 1 - \mathcal{F}_{\mathfrak{b},\overline{\boldsymbol{\omega}}}\left(\frac{c_1}{b_1}\right) = 1 - \mathcal{F}_{\mathfrak{b},\overline{\boldsymbol{\omega}}}(x).$$

Now suppose that (3) holds for N,

$$\mathcal{F}_{\mathfrak{b},\omega}\Bigl(1-\sum_{n=1}^{N}\frac{e_n}{\prod_{k=1}^{n}b_k}\Bigr)=1-\mathcal{F}_{\mathfrak{b},\overline{\omega}}\Bigl(\sum_{n=1}^{N}\frac{e_n}{\prod_{k=1}^{n}b_k}\Bigr)$$

for every possible sequence $\{e_n\}_{n=1}^{\infty}$. Then using Lemma 1 we obtain for $x=\sum_{n=1}^{N+1}\frac{c_n}{\prod_{k=1}^{n}b_k}$ that

$$\begin{aligned}
&\mathcal{F}_{\mathfrak{b},\omega}(1-x)\\
&=\mathcal{F}_{\mathfrak{b},\omega}\Bigl(\frac{b_1-c_1-1}{b_1}+\frac{1}{b_1}\Bigl(\sum_{n=1}^{N-1}\frac{b_{n+1}-c_{n+1}-1}{\prod_{k=1}^{n}b_{k+1}}+\frac{b_{(N-1)+1}-c_{(N-1)+1}}{\prod_{k=1}^{N-1}b_{k+1}}\Bigr)\Bigr)\\
&=f_1(b_1-c_1-1)+\Delta_1(b_1-c_1-1)\mathcal{F}_{\mathfrak{b}^{(1)},\omega^{(1)}}(1-x^{(1)})\\
&=f_1(b_1-c_1-1)+\Delta_1(b_1-c_1-1)\bigl(1-\mathcal{F}_{\mathfrak{b}^{(1)},\overline{\omega}^{(1)}}(x^{(1)})\bigr)\\
&=1-\overline{f}_1(c_1+1)+\overline{\Delta}_1(c_1)\bigl(1-\mathcal{F}_{\mathfrak{b}^{(1)},\overline{\omega}^{(1)}}(x^{(1)})\bigr)\\
&=1-\bigl(\overline{f}_1(c_1)+\overline{\Delta}_1(c_1)\mathcal{F}_{\mathfrak{b}^{(1)},\overline{\omega}^{(1)}}(x^{(1)})\bigr)=1-\mathcal{F}_{\mathfrak{b},\overline{\omega}}(x)\,.
\end{aligned}$$

■

Remark 1 *One can prove that if* $\prod_{k=1}^{\infty}\max_{c=0,\dots,b_k-1}\Delta_k(c)=0$ *then* $\mathcal{F}_{\mathfrak{b},\omega}$ *is continuous on the interval* $[0,1]$. *One can then extend Lemma 4 for every* $x\in[0,1]$.

Lemma 5 (Difference) *For every* $N\in\mathbb{N}$

$$\mathcal{F}_{\mathfrak{b},\omega}\Bigl(\sum_{n=1}^{N-1}\frac{c_n}{\prod_{k=1}^{n}b_k}+\frac{c_N+1}{\prod_{k=1}^{N}b_k}\Bigr)-\mathcal{F}_{\mathfrak{b},\omega}\Bigl(\sum_{n=1}^{N}\frac{c_n}{\prod_{k=1}^{n}b_k}\Bigr)=\prod_{k=1}^{N}\Delta_k(c_k)\,. \quad (4)$$

Proof. Denote the left-hand side of (4) by LHS. Then if $c_N\le b_N-2$ then

$$\begin{aligned}
\text{LHS}&=\prod_{k=1}^{N-1}\Delta_k(c_k)\cdot\Bigl(\mathcal{F}_{\mathfrak{b}^{(N-1)},\omega^{(N-1)}}\Bigl(\frac{c_N+1}{b_N}\Bigr)-\mathcal{F}_{\mathfrak{b}^{(N-1)},\omega^{(N-1)}}\Bigl(\frac{c_N}{b_N}\Bigr)\Bigr)\\
&=\prod_{k=1}^{N-1}\Delta_k(c_k)\cdot\bigl(f_N(c_N+1)-f_N(c_N)\bigr)=\prod_{k=1}^{N}\Delta_k(c_k)\,.
\end{aligned}$$

In the case $c_N = b_N - 1$ we apply the first case on the function $\mathcal{F}_{\mathfrak{b},\overline{\omega}}$,

$$\begin{aligned}
\text{LHS} &= \left(1 - \mathcal{F}_{\mathfrak{b},\overline{\omega}}\Big(\sum_{n=1}^{N-1} \frac{b_n - c_n - 1}{\prod_{k=1}^{n} b_k}\Big)\right) \\
&\quad - \left(1 - \mathcal{F}_{\mathfrak{b},\overline{\omega}}\Big(\sum_{n=1}^{N-1} \frac{b_n - c_n - 1}{\prod_{k=1}^{n} b_k} + \frac{1}{\prod_{k=1}^{N} b_k}\Big)\right) \\
&= \prod_{k=1}^{N} \overline{\Delta}_k(b_n - c_n - 1) = \prod_{k=1}^{N} \Delta_k(c_k)\,.
\end{aligned}$$

■

In the following text we will use the symbol

$$\Theta_k(c) := b_k \Delta_k(c)\,.$$

Lemma 6 (Derivative) *Let* $x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k} \in (0,1)$. *Suppose that the derivative* $\mathcal{F}'_{\mathfrak{b},\omega}(x)$ *exists and is finite. Then*

$$\mathcal{F}'_{\mathfrak{b},\omega}(x) = \prod_{k=1}^{\infty} \Theta_k(c_k)\,. \tag{5}$$

In particular, this product has a finite value.

Proof. We have

$$\lim_{N\to\infty} \frac{\mathcal{F}_{\mathfrak{b},\omega}\Big(\sum_{n=1}^{N-1} \frac{c_n}{\prod_{k=1}^{n} b_k} + \frac{c_N + 1}{\prod_{k=1}^{N} b_k}\Big) - \mathcal{F}_{\mathfrak{b},\omega}\Big(\sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}\Big)}{\Big(\sum_{n=1}^{N-1} \frac{c_n}{\prod_{k=1}^{n} b_k} + \frac{c_N + 1}{\prod_{k=1}^{N} b_k}\Big) - \Big(\sum_{n=1}^{N} \frac{c_n}{\prod_{k=1}^{n} b_k}\Big)} \tag{6}$$

$$= \lim_{N\to\infty} \left(\frac{\sum_{n=1}^{N-1} \frac{c_n}{\prod_{k=1}^{n} b_k} + \frac{c_N + 1}{\prod_{k=1}^{N} b_k} - x}{\frac{1}{\prod_{k=1}^{N} b_k}} \right. \tag{7}$$

$$\cdot \frac{\mathcal{F}_{\mathfrak{b},\omega}\Big(\sum\limits_{n=1}^{N-1}\frac{c_n}{\prod_{k=1}^{n} b_k}+\frac{c_N+1}{\prod_{k=1}^{N} b_k}\Big)-\mathcal{F}_{\mathfrak{b},\omega}(x)}{\sum\limits_{n=1}^{N-1}\frac{c_n}{\prod_{k=1}^{n} b_k}+\frac{c_N+1}{\prod_{k=1}^{N} b_k}-x} \tag{8}$$

$$+\frac{x-\sum\limits_{n=1}^{N}\frac{c_n}{\prod_{k=1}^{n} b_k}}{\frac{1}{\prod_{k=1}^{N} b_k}}\cdot\frac{\mathcal{F}_{\mathfrak{b},\omega}(x)-\mathcal{F}_{\mathfrak{b},\omega}\Big(\sum\limits_{n=1}^{N}\frac{c_n}{\prod_{k=1}^{n} b_k}\Big)}{x-\sum\limits_{n=1}^{N}\frac{c_n}{\prod_{k=1}^{n} b_k}}\Bigg) \tag{9}$$

$$=\mathcal{F}'_{\mathfrak{b},\omega}(x)\lim_{N\to\infty}\frac{\frac{1}{\prod_{k=1}^{N} b_k}}{\frac{1}{\prod_{k=1}^{N} b_k}}=\mathcal{F}'_{\mathfrak{b},\omega}(x)\,.$$

Existence of $\mathcal{F}'_{\mathfrak{b},\omega}(x)$ implies that limits of (8) and of the second fraction in (9) are equal to $\mathcal{F}'_{\mathfrak{b},\omega}(x)$. Hence the limit (6) exists and is equal to $\mathcal{F}'_{\mathfrak{b},\omega}(x)$. In the case that $x=\sum\limits_{n=1}^{N}\frac{c_n}{\prod_{k=1}^{n} b_k}$ we obtain that (6) $=\mathcal{F}'_{\mathfrak{b},\omega}(x)$ immediately.

On the other hand, Lemma 5 implies that

$$(6)=\lim_{N\to\infty}\frac{\prod\limits_{k=1}^{N}\Delta_k(c_k)}{\frac{1}{\prod_{k=1}^{N} b_k}}=\prod_{k=1}^{\infty}\Theta_k(c_k)\,.$$

■

Corollary 1 *For almost every* $x\in[0,1]$ *the derivative* $\mathcal{F}'_{\mathfrak{b},\omega}(x)$ *exists and is finite. In particular, for almost every* $x=\sum\limits_{n=1}^{\infty}\frac{c_n}{\prod_{k=1}^{n} b_k}$ *the product* $\prod_{n=1}^{\infty}\Theta_k(c_k)$ *exists and is finite (possibly zero).*

Proof. The function $\mathcal{F}_{\mathfrak{b},\omega}$ is bounded and nondecreasing, hence in almost all points it has a finite derivative. According to Lemma 6 we obtain that the product (5) is finite. ■

3 Main result

Our main result is a proof of Theorem 1.

Proof. A number $x \in (0,1)$ is not absolutely normal if there exist $b \geq 2$, $L \in \mathbb{N}$ and $a_1, \ldots, a_L \in \{0, \ldots, b-1\}$ such that if $x = \sum_{n=1}^{\infty} \frac{c_n}{b^n}$ then

$$\liminf_{n\to\infty} \frac{\#\{n \leq N-L \mid c_{n+i} = a_i, i = 1, \ldots, L\}}{N} < \frac{1}{b^L}.$$

Then there exists $s \in \{0, \ldots, L-1\}$ such that

$$\liminf_{n\to\infty} \frac{\#\{n \leq N-L, n \equiv s \ (\mathrm{mod} L) \mid c_{n+i} = a_i, i = 1, \ldots, L\}}{N} < \frac{1}{Lb^L}.$$

Hence for some rational $\beta < \frac{1}{Lb^L}$

$$\liminf_{n\to\infty} \frac{\#\{n \leq N-L, n \equiv s \ (\mathrm{mod} L) \mid c_{n+i} = a_i, i = 1, \ldots, L\}}{N} \leq \beta. \tag{10}$$

Denote by $R_{b,L,a,s,\beta}$ the set of all $x = \sum_{n=1}^{\infty} \frac{c_n}{b^n}$ satisfying (10). The result of the previous paragraph is that the set of not absolutely normal numbers is a subset of

$$\bigcup_{b=2}^{\infty} \bigcup_{L=1}^{\infty} \bigcup_{a_1,\ldots,a_L=0}^{b-1} \bigcup_{s=0}^{L-1} \bigcup_{\beta \in (0, \frac{1}{Lb^L}) \cap \mathbb{Q}} R_{b,L,a,s,\beta}.$$

It is sufficient to prove that every set $R_{b,L,a,s,\beta}$ has zero measure. Then the set of not absolutely normal numbers is a subset of a countable union of null sets, hence it is a null set.

Let $b \geq 2$, $L \in \mathbb{N}$, $a_1, \ldots, a_L \in \{0, \ldots, b-1\}$, $s \in \{0, \ldots, L-1\}$ and $\beta \in \left(0, \frac{1}{Lb^L}\right)$. Put $A = a_1 b^{L-1} + a_2 b^{L-2} + \cdots + a_L$. Let

$$x = \sum_{n=1}^{\infty} \frac{c_n}{b^n} = \sum_{n=1}^{s} \frac{d_n}{b^n} + \frac{1}{b^s} \sum_{n=1}^{\infty} \frac{d_{s+n}}{b^{Ln}} \in R_{b,L,a,s,\beta}.$$

Then obviously,

$$\#\big\{n \leq N-L, n \equiv s \ (\mathrm{mod} L) \ \big| \ c_{n+i} = a_i, i = 1, \ldots, L\big\} \\ = \#\Big\{s < n \leq \Big[\frac{N-s}{L}\Big] \ \Big| \ d_n = A\Big\}.$$

Hence

$$\liminf_{M\to\infty} \frac{\#\{s < n \leq M \mid d_n = A\}}{M} = \liminf_{N\to\infty} \frac{\#\big\{s < n \leq \big[\frac{N-s}{L}\big] \ \big| \ d_n = A\big\}}{\big[\frac{N-s}{L}\big]}$$

$$
\begin{aligned}
&= \liminf_{N\to\infty} \frac{N}{\left[\frac{N-s}{L}\right]} \frac{\#\left\{n \le N-L, n \equiv s \ (\mathrm{mod} L) \ \middle| \ c_{n+i} = a_i, i = 1, \dots, L\right\}}{N} \\
&\le L\beta\,.
\end{aligned}
$$

From this we obtain that $R_{b,L,\mathbf{a},s,\beta} \subseteq P$, where

$$
P = \left\{x = \sum_{n=1}^{s} \frac{d_n}{b^n} + \frac{1}{b^s} \sum_{n=1}^{\infty} \frac{d_{s+n}}{b^{Ln}} \ \middle| \ \liminf_{N\to\infty} \frac{\#\{s < n \le N \mid d_n = A\}}{N} \le L\beta\right\}.
$$

Thus it is sufficient to prove that the set P has zero measure.

Let $\alpha \in \left(L\beta, \frac{1}{b^L}\right)$. For $t \in [0,1]$ define

$$
\varphi_\alpha(t) := t^\alpha \left(\frac{b^L - t}{b^L - 1}\right)^{1-\alpha}.
$$

The function φ_α is continuous with $\varphi_\alpha(0) = 0$, $\varphi_\alpha(1) = 1$ and $\varphi'_\alpha(1) = \dfrac{\alpha b^L - 1}{b^L - 1} < 0$. Hence there is $T \in (0,1)$ with $\varphi_\alpha(T) = 1$. For $u \in (0,1)$ put

$$
\psi(u) := \varphi_u(T) = T^u \left(\frac{b^L - T}{b^L - 1}\right)^{1-u}.
$$

The function ψ is continuous and decreasing with $\psi(\alpha) = 1$.

Consider the function $\mathcal{F}_{\mathbf{b},\boldsymbol{\omega}}$ corresponding to $\mathbf{b} = \{b_k\}_{k=1}^{\infty}$ with

$$
b_k = \begin{cases} b, & \text{if } k \le s, \\ b^L, & \text{if } k > s, \end{cases}
$$

and $\boldsymbol{\omega} = \{\omega_k\}_{k=1}^{\infty}$ with

$$
\Delta_k(d) = \begin{cases} \frac{1}{b}, & \text{if } k \le s, \\ \frac{T}{b^L}, & \text{if } k > s \text{ and } d = A, \\ \frac{b^L - T}{b^L(b^L - 1)}, & \text{if } k > s \text{ and } d \ne A. \end{cases}
$$

We have

$$
\Theta_k(d) = \begin{cases} 1, & \text{if } k \le s\,, \\ T, & \text{if } k > s \text{ and } d = A, \\ \frac{b^L - T}{b^L - 1}, & \text{if } k > s \text{ and } d \ne A. \end{cases}
$$

Now Corollary 1 implies that for almost every $x = \sum_{n=1}^{s} \frac{d_n}{b^n} + \frac{1}{b^s} \sum_{n=1}^{\infty} \frac{d_{n+s}}{b^{Ls}}$ the following product exists and is finite

$$\prod_{n=1}^{\infty} \Theta_n(d_n) = \lim_{N\to\infty} T^{\#\{s<n\leq N \mid d_n=A\}} \Big(\frac{b^L - T}{b^L - 1}\Big)^{N-\#\{s<n\leq N \mid d_n=A\}}$$

$$= \lim_{N\to\infty} \Big(\psi\Big(\frac{\#\{s < n \leq N \mid d_n = A\}}{N}\Big)\Big)^N. \tag{11}$$

Now suppose that $x \in P$. Then

$$\limsup_{N\to\infty} \psi\Big(\frac{\#\{s < n \leq N \mid d_n = A\}}{N}\Big) = \psi\Big(\liminf_{N\to\infty} \frac{\#\{s < n \leq N \mid d_n = A\}}{N}\Big)$$
$$\geq \psi(L\beta) > \psi(\alpha) = 1,$$

hence

$$\limsup_{N\to\infty} \Big(\psi\Big(\frac{\#\{s < n \leq N \mid d_n = A\}}{N}\Big)\Big)^N = \infty,$$

contradicting finiteness of (11). Thus the set P has zero measure. ■

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