LINEAR RECURSION RELATIONS

Method of Characteristic Equation:

Example 1: Find a general expression for x_n satisfying

$$x_{n+1} + \alpha x_n + \beta x_{n-1} = 0 \quad , \qquad n = 1, 2, 3, \ldots \; ; \; \alpha, \beta \text{ constants} \tag{1}$$
$$x_0 \text{ and } x_1 \text{ given.} \tag{2}$$

Solution: (Usually the preferred approach): Start by ignoring (2), and ask if we can find any two linearly independent solutions to (1). If we can, we can then form a linear combination of the two that satisfies (2), finishing the task. It turns out to be a good idea to try solutions of the form

$$x_n = r^n. \tag{3}$$

Substituting (3) into (1) gives

$$r^{n+1} + \alpha r^n + \beta r^{n-1} = 0, \text{ i.e. the } \textit{Characteristic equation} \text{ is } \; r^2 + \alpha r + \beta = 0.$$

A quadratic equation usually has two distinct solutions r_1 and r_2 (which may be complex). Since (1) is a linear relation, it will be satisfied by

$$x_n = c_1 r_1^n + c_2 r_2^n \tag{4}$$

for any choice of the two constants c_1 and c_2. We next choose the constants so that (2) is also satisfied. That gives a closed form expression satisfying both (1) and (2). Since the combination of (1) and (2) clearly has a unique solution, that one must be what we have found.

If there is a double root $r_1 = r_2$, then the general form of the solution will not be as given by (4) but instead $x_n = c_1 r_1^n + c_2 n r_1^n$ (continuing with $c_3 n^2 r_1^n$, etc. in cases of recursions with roots of still higher multiplicities).

Example 2: Find a general expression for x_n satisfying the *inhomogeneous* linear recursion relation

$$x_n + 5 x_{n-1} = \frac{1}{n} \quad , \qquad n = 1, 2, 3, \ldots \tag{5}$$
$$x_0 = \ln 6 - \ln 5. \tag{6}$$

Note: Equations (5) and (6) arise from the relation $x_n = \int_0^1 \frac{t^n}{t+5} dt$.

Solution: Inhomogeneous linear difference equations can often be solved by *variation of parameters*, very much like the case for inhomogeneous linear differential equations. We first find the *general solution* to the homogeneous equation: $x_n + 5 x_{n-1} = 0 \;\Rightarrow\; x_n = c(-5)^n$. Next, replace the arbitrary constant c by another function of n, say q_n , i.e. $x_n = q_n (-5)^n$, and substitute into (5). This gives , after a brief simplification, $q_n = q_{n-1} + \frac{1}{n(-5)^n}$, for which we immediately find one (which is enough) solution $q_n = \sum_{k=1}^n \frac{1}{k(-5)^k}$. The general solution can now be written as the sum of the general solution to the homogeneous equation and one particular solution, i.e. $x_n = (-5)^n \left[c + \sum_{k=1}^n \frac{1}{k(-5)^k} \right]$. As usual, the initial condition can be used to determine the constant; in the present case giving $c = x_0 = \ln 6 - \ln 5$. We can note that, this value of $c \,(= x_0)$ is the only one for which x_n does not diverge when n increases.

Method of Generating Function:

We consider again the problem given in Example 1.

Solution: We define a generating function $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} x_n \, z^n \, . \tag{7}$$

It will then hold that

$$\begin{aligned} f(z) &= x_0 z^0 + x_1 z^1 + x_2 z^2 + x_3 z^3 + \dots \quad , \\ \alpha z f(z) &= \alpha x_0 z^1 + \alpha x_1 z^2 + \alpha x_2 z^3 + \dots \quad , \\ \beta z^2 f(z) &= \beta x_0 z^2 + \beta x_1 z^3 + \dots \quad . \end{aligned}$$

When adding these, all terms in the RHS beyond the first two vanish because of (1). Therefore

$$(1 + \alpha z + \beta z^2) f(z) = x_0 + (x_1 + \alpha x_0) z \quad ,$$

i.e.

$$f(z) = \frac{x_0 + (x_1 + \alpha x_0) z}{1 + \alpha z + \beta z^2} . \tag{8}$$

In order to get the Taylor expansion of $f(z)$ from (8) (i.e. the elements of the original recursion according to (7)), we expand (8) into partial fractions and then add the Taylor expansions for each of the fractions (which are trivially obtained after noting that each fraction represents the sum of a geometric progression in powers of z).

We can note that the quadratic equation we need to solve in order to do the partial fraction decomposition is just the same as the one that arose in the characteristic equation approach. We will arrive at the same solution.

The following is a generating function solution to Example 2:

Solution: Let

$$f(z) = x_0 + x_1 z + x_2 z^2 + x_3 z^3 + \dots \quad , \tag{9}$$

i.e.

$$5z\, f(z) = 5x_0 z + 5x_1 z^2 + 5x_2 z^3 + \dots \quad .$$

Adding the equations and utilizing the recursion formula gives:

$$(1 + 5z) f(z) = x_0 + \tfrac{1}{1} z + \tfrac{1}{2} z^2 + \tfrac{1}{3} z^3 + \dots$$

Next, divide both sides by $(1 + 5z)$ and further note that

$$\frac{1}{1 + 5z} = 1 + (-5) z + (-5)^2 z^2 + (-5)^3 z^3 + \dots$$

Therefore

$$\begin{aligned} f(z) &= \left(1 + (-5) z + (-5)^2 z^2 + (-5)^3 z^3 + \dots\right)\left(x_0 + \frac{1}{1} z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \dots\right) = \\ &= \quad x_0 + (-5) x_0 \, z + (-5)^2 x_0 \, z^2 + (-5)^3 x_0 \, z^3 + \dots \\ &\quad + (-5)^1 \left(\sum_{k=1}^{1} \frac{1}{k\,(-5)^k} \right) z + (-5)^2 \left(\sum_{k=1}^{2} \frac{1}{k\,(-5)^k} \right) z^2 + (-5)^3 \left(\sum_{k=1}^{3} \frac{1}{k\,(-5)^k} \right) z^3 + \dots \end{aligned}$$

Equating coefficients between the last line above and (9) gives

$$x_n = (-5)^n \left[x_0 + \sum\nolimits_{k=1}^{n} \frac{1}{k\,(-5)^k} \right] ,$$

which is exactly the same result as we obtained via the *characteristic equation* and *variation of parameters*.