LEHMER'S METHOD FOR LOCATING ZERO'S OF POLYNOMIALS

Lehmer's method is normally formulated as a technique to determine whether a polynomial $f_0(z)$has any roots *inside the unit circle*. A trivial modification (a change of variable $z = 1/x$, followed by multiplying by x^n - to obtain a polynomial in x) turns the method into to a test for zeros *outside the unit circle*.

Let the polynomial to be tested be

$$f_0(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

where the coefficients may be complex numbers (with neither a_n or a_0 equal to zero). Introduce next a polynomial with the same coefficients, but conjugated (if complex) and in reversed order:

$$g_0(z) = \overline{a_0} z^n + \overline{a_1} z^{n-1} + \ldots + \overline{a_{n-1}} z + \overline{a_n} \qquad (= z^n \, \overline{f_0}(\bar{z}^{-1}))$$

Next form

$$f_1(z) = \overline{a_0} \cdot f_0(z) - a_n \cdot g_0(z)$$ (this is of at least one degree less than $f_0(z)$)

$$g_1(z) = z^m \, \overline{f_1}(\bar{z}^{-1})$$ (same coefficient reversal as above; m is the degree of $f_1(z)$; typically $m = n - 1$),

and proceed like this (obtaining $f_2(z)$, $g_2(z)$, ...) **until the first occurrence of $f_k(0) \le 0$ for $k > 0$** (i.e. we are testing the constant term of the polynomials $f_k(z)$). The following theorem holds:

Theorem:

- **if $f_k(0)$ *is negative* , then $f_0(z)$ has at least one zero inside the unit circle,**
- **if $f_k(0)$ *is zero* and $f_{k-1}(z)$ is equal to a constant, then $f_0(z)$ has no zero inside the unit circle.**

Example 1: Test $x^3 - 2x^2 - 4x + 8 = 0$ for zeros inside the unit circle.

We get

$$\begin{aligned}
&f_0(x) = x^3 - 2x^2 - 4x + 8 && g_0(x) = 8x^3 - 4x^2 - 2x + 1 \\
&f_1(x) = -12x^2 - 30x + 63 && g_1(x) = 63x^2 - 30x - 12 \\
&f_2(x) = -2250x + 3825 && g_2(x) = 3825x - 2250 \\
&f_3(x) = 9568125 && g_3(x) = 9568125 \\
&f_4(x) = 0 && \text{- - - - - - - - - - - - -}
\end{aligned}$$

Here $f_4(0) = 0$, the signs for $f_1(0)$, $f_2(0)$, $f_3(0)$ are all positive, and $f_3(x)$ is equal to a constant. Therefore, we can conclude that $f_0(x)$ has no zero inside the unit circle.

Example 2: The third order BDF scheme for ODEs can be written

$$-\tfrac{11}{6}y_{n+1} + 3y_n - \tfrac{3}{2}y_{n-1} + \tfrac{1}{3}y_{n-2} = y'_{n+1} \ .$$

Test for roots of the characteristic polynomial *outside* the unit circle, and determine the stability of the method.

We get $\rho(r) = -\tfrac{11}{6}r^3 + 3r^2 - \tfrac{3}{2}r + \tfrac{1}{3}$, and therefore test $f_0(x) = x^3\,\rho(1/x) = \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 3x - \tfrac{11}{6}$ for roots inside the unit circle. This gives

$$\begin{array}{lll}
f_0(x) = \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 3x - \tfrac{11}{6} & g_0(x) = -\tfrac{11}{6}x^3 + 3x^2 - \tfrac{3}{2}x + \tfrac{1}{3} & (\ = \ \rho(x)\,) \\
f_1(x) = \tfrac{7}{4}x^2 - 5x + \tfrac{13}{4} & g_1(x) = \tfrac{13}{4}x^2 - 5x + \tfrac{7}{4} & \\
f_2(x) = -\tfrac{15}{2}x + \tfrac{15}{2} & g_2(x) = \tfrac{15}{2}x - \tfrac{15}{2} & \\
f_3(x) = 0 & \text{- - - - - - - - -} &
\end{array}$$

Here $f_3(0) = 0$, and the signs for $f_1(0)$, $f_2(0)$ are both positive. However, $f_2(x)$ is not equal to a constant, so we have encountered the 'loophole' in the theorem. This is related to the (for ODE methods known) root at $r = 1$ - this is located on the unit circle. Dividing away this root before applying Lehmer's method gives

$$\begin{array}{ll}
f_0(x) = \tfrac{1}{3}x^2 - \tfrac{7}{6}x + \tfrac{11}{6} & g_0(x) = \tfrac{11}{6}x^2 - \tfrac{7}{6}x + \tfrac{1}{3} \\
f_1(x) = -\tfrac{7}{4}x + \tfrac{13}{4} & g_1(x) = \tfrac{13}{4}x - \tfrac{7}{4} \\
f_2(x) = \tfrac{15}{2} & g_2(x) = \tfrac{15}{2} \\
f_3(x) = 0 & \text{- - - - - - - - -}
\end{array}$$

Again, $f_3(0) = 0$, and the signs for $f_1(0)$, $f_2(0)$ are both positive. This time $f_2(x)$ is equal to a constant. We have therefore no root outside the unit circle.

There is still a possibility that the BDF method could be unstable due to a multiple root on the unit circle. A completely general way to test for multiple roots goes as follows:

To test for multiple roots of $f_0(x)$, define $f_1(x) = f_0'(x)$. The sequence

$$f_k(x) = \{\text{remainder of 'long division'} \quad f_{k-2}(x)\,/f_{k-1}(x)\} \quad , \ k=2,3,...$$

forms polynomials of decreasing degrees, however still preserving as roots any which are shared by $f_{k-2}(x)$ and $f_{k-1}(x)$, i.e. between $f_0(x)$ and $f_1(x)$ - in this case the multiple roots of $f_0(x)$. We find these by inspecting the last non-vanishing polynomial in the sequence - if this polynomial is a constant, there were no common roots of $f_0(x)$ and $f_1(x)$.

Testing $f_0(x) = \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 3x - \tfrac{11}{6}$ for multiple roots gives

$$\begin{array}{l}
f_1(x) = x^2 - 3x + 3 \\
f_2(x) = \tfrac{1}{2}x - \tfrac{1}{3} \\
f_3(x) = \tfrac{13}{9} \\
f_4(x) = 0
\end{array}$$

The last non-vanishing polynomial is a constant; hence $f_0(x)$ has no multiple roots.

In conclusion, the BDF scheme of order 3 is stable.

Example 3: The seventh order BDF scheme for ODEs can be written

$$-\tfrac{363}{140}y_{n+1}+7y_n-\tfrac{21}{2}y_{n-1}+\tfrac{35}{3}y_{n-2}-\tfrac{35}{4}y_{n-3}+\tfrac{21}{5}y_{n-4}-\tfrac{7}{6}y_{n-5}+\tfrac{1}{7}y_{n-6}=y'_{n+1}$$

Test for roots of the characteristic polynomial *outside* the unit circle, and determine the stability of the method.

We get $\rho(r)=-\tfrac{363}{140}r^7+7r^6-\tfrac{21}{2}r^5+\tfrac{35}{3}r^4-\tfrac{35}{4}r^3+\tfrac{21}{5}r^2-\tfrac{7}{6}r+\tfrac{1}{7}$, and form the sequence of polynomials $f_0(x)=\tfrac{1}{7}x^7-\tfrac{7}{6}x^6+\tfrac{21}{5}x^5-\tfrac{35}{4}x^4+\tfrac{35}{3}x^3-\tfrac{21}{2}x^2+7x-\tfrac{363}{140}$,

$$g_0(x)=-\tfrac{363}{140}r^7+7r^6-\tfrac{21}{2}r^5+\tfrac{35}{3}r^4-\tfrac{35}{4}r^3+\tfrac{21}{5}r^2-\tfrac{7}{6}r+\tfrac{1}{7},$$

and

$\{ f_k(x), g_k(x),\ k = 1,2,\ldots \}$ - until $f_k(0) \le 0$.

Straightforward algebra gives in this case the following signs:

$f_1(0)$	$f_2(0)$	$f_3(0)$	$f_4(0)$	$f_5(0)$	...
+	+	+	+	-	

We can stop at this point; a negative sign occurred in the sequence. The polynomial $f_0(x)$ has therefore at least one zero inside the unit circle. As a consequence, the BDF method of order seven is unstable.