Lehmer's method is normally formulated as a technique to determine whether a polynomial \( f_0(z) \) has any roots inside the unit circle. A trivial modification (a change of variable \( z = 1/x \), followed by multiplying by \( x^n \) to obtain a polynomial in \( x \)) turns the method into a test for zeros outside the unit circle.

Let the polynomial to be tested be

\[
f_0(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0
\]

where the coefficients may be complex numbers (with neither \( a_n \) or \( a_0 \) equal to zero). Introduce next a polynomial with the same coefficients, but conjugated (if complex) and in reversed order:

\[
g_0(z) = \overline{a_0} z^n + \overline{a_1} z^{n-1} + \ldots + \overline{a_{n-1}} z + \overline{a_n} \quad (= z^n \overline{f_0(\overline{z}^{-1})})
\]

Next form

\[
f_1(z) = \overline{a_0} \cdot f_0(z) - a_n \cdot g_0(z) \quad \text{(this is of at least one degree less than } f_0(z) \text{)}
\]

\[
g_1(z) = z^m \overline{f_1(\overline{z}^{-1})} \quad \text{(same coefficient reversal as above; } m \text{ is the degree of } f_1(z); \text{ typically } m = n - 1),
\]

and proceed like this (obtaining \( f_2(z), g_2(z), \ldots \)) until the first occurrence of \( f_k(0) \leq 0 \) for \( k > 0 \) (i.e. we are testing the constant term of the polynomials \( f_k(z) \)). The following theorem holds:

**Theorem:**

- If \( f_k(0) \) is negative, then \( f_0(z) \) has at least one zero inside the unit circle,
- If \( f_k(0) \) is zero and \( f_{k+1}(z) \) is equal to a constant, then \( f_0(z) \) has no zero inside the unit circle.

**Example 1:** Test \( x^3 - 2x^2 - 4x + 8 = 0 \) for zeros inside the unit circle.

We get

\[
\begin{align*}
f_0(x) &= x^3 - 2x^2 - 4x + 8 \quad &g_0(x) &= 8x^3 - 4x^2 - 2x + 1 \\
f_1(x) &= -12x^2 - 30x + 63 \quad &g_1(x) &= 63x^2 - 30x - 12 \\
f_2(x) &= -2250x + 3825 \quad &g_2(x) &= 3825x - 2250 \\
f_3(x) &= 9568125 \quad &g_3(x) &= 9568125 \\
f_4(x) &= 0
\end{align*}
\]

Here \( f_4(0) = 0 \), the signs for \( f_1(0), f_2(0), f_3(0) \) are all positive, and \( f_3(x) \) is equal to a constant. Therefore, we can conclude that \( f_0(x) \) has no zero inside the unit circle.
Example 2: The third order BDF scheme for ODEs can be written

\[-\frac{11}{6} y_{n+1} + 3y_n - \frac{3}{2} y_{n-1} + \frac{1}{3} y_{n-2} = y'_{n+1}.\]

Test for roots of the characteristic polynomial outside the unit circle, and determine the stability of the method.

We get \(\rho(r) = -\frac{11}{6} r^3 + 3r^2 - \frac{3}{2} r + \frac{1}{3}\), and therefore test \(f_0(x) = x^3 \rho(1/x) = \frac{1}{3} x^3 - \frac{3}{2} x^2 + 3x - \frac{11}{6}\) for roots inside the unit circle. This gives

\[
\begin{align*}
    f_0(x) &= \frac{1}{3} x^3 - \frac{3}{2} x^2 + 3x - \frac{11}{6} \\
    f_1(x) &= \frac{7}{4} x^2 - 5x + \frac{13}{4} \\
    f_2(x) &= -\frac{15}{2} x + \frac{15}{2} \\
    f_3(x) &= 0
\end{align*}
\]

Here \(f_3(0) = 0\), and the signs for \(f_1(0), f_2(0)\) are both positive. However, \(f_3(x)\) is not equal to a constant, so we have encountered the 'loophole' in the theorem. This is related to the (for ODE methods known) root at \(r = 1\) - this is located on the unit circle. Dividing away this root before applying Lehmer's method gives

\[
\begin{align*}
    f_0(x) &= \frac{1}{3} x^2 - \frac{7}{6} x + \frac{11}{6} \\
    f_1(x) &= -\frac{7}{4} x + \frac{13}{4} \\
    f_2(x) &= \frac{15}{2} \\
    f_3(x) &= 0
\end{align*}
\]

Again, \(f_3(0) = 0\), and the signs for \(f_1(0), f_2(0)\) are both positive. This time \(f_3(x)\) is equal to a constant. We have therefore no root outside the unit circle.

There is still a possibility that the BDF method could be unstable due to a multiple root on the unit circle. A completely general way to test for multiple roots goes as follows:

To test for multiple roots of \(f_0(x)\), define \(f_i(x) = f_i'(x)\). The sequence

\[f_k(x) = \text{remainder of 'long division' } f_0(x) f_k(x), \quad k=2,3,...\]

forms polynomials of decreasing degrees, however still preserving as roots any which are shared by \(f_k(x)\) and \(f_0(x)\), i.e. between \(f_0(x)\) and \(f_1(x)\) - in this case the multiple roots of \(f_0(x)\). We find these by inspecting the last non-vanishing polynomial in the sequence - if this polynomial is a constant, there were no common roots of \(f_0(x)\) and \(f_1(x)\).

Testing

\[
\begin{align*}
    f_0(x) &= \frac{1}{3} x^3 - \frac{3}{2} x^2 + 3x - \frac{11}{6} \\
    f_1(x) &= x^2 - 3x + 3 \\
    f_2(x) &= \frac{1}{2} x - \frac{3}{2} \\
    f_3(x) &= \frac{13}{9} \\
    f_4(x) &= 0
\end{align*}
\]

The last non-vanishing polynomial is a constant; hence \(f_0(x)\) has no multiple roots.

In conclusion, the BDF scheme of order 3 is stable.
Example 3: The seventh order BDF scheme for ODEs can be written

\[
-\frac{363}{140}y_{n+1} + 7y_n - \frac{21}{2}y_{n-1} + \frac{35}{3}y_{n-2} - \frac{35}{4}y_{n-3} + \frac{21}{5}y_{n-4} - \frac{7}{6}y_{n-5} + \frac{7}{7}y_{n-6} = y_{n+1}
\]

Test for roots of the characteristic polynomial outside the unit circle, and determine the stability of the method.

We get \( \rho(r) = -\frac{363}{140}r^7 + 7r^6 - \frac{21}{2}r^5 + \frac{35}{3}r^4 - \frac{35}{4}r^3 + \frac{21}{5}r^2 - \frac{7}{6}r + \frac{7}{7} \), and form the sequence of polynomials

\[
f_0(x) = \frac{363}{140}x^7 - \frac{7}{6}x^6 + \frac{21}{5}x^5 - \frac{35}{4}x^4 + \frac{35}{3}x^3 - \frac{21}{5}x^2 + 7x - \frac{363}{140},
\]

\[
g_0(x) = \frac{363}{140}x^7 + 7x^6 - \frac{21}{2}x^5 + \frac{35}{3}x^4 - \frac{35}{4}x^3 + \frac{21}{5}x^2 - \frac{7}{6}x + \frac{7}{7},
\]

and

\[
\{ f_k(x), g_k(x), k = 1,2,... \} \quad \text{until} \quad f_k(0) \leq 0.
\]

Straightforward algebra gives in this case the following signs:

\[
f_1(0) \quad f_2(0) \quad f_3(0) \quad f_4(0) \quad f_5(0) \quad \ldots
\]

+ \quad + \quad + \quad + \quad -

We can stop at this point; a negative sign occurred in the sequence. The polynomial \( f_6(x) \) has therefore at least one zero inside the unit circle. As a consequence, the BDF method of order seven is unstable.