Lecture #21

1 We now consider the trigonometric interpolation problem. We seek for a function of the following form

$$p(x) = a_0 + \sum_{j=1}^{n} a_j \cos(jx) + b_j \sin(jx).$$

If a_n and b_n are not both zero, then this is called a trigonometric polynomial of degree n.

We've implicitly given a basis for our function space (dimension $2n+1$)

$$\{1, \cos(x), \sin(x), \ldots, \cos(nx), \sin(nx)\}.$$

We should really first consider whether these functions are linearly independent; otherwise the interpolation problem posed here might have multiple solutions (if it has any). I will assume you already know that these functions are a basis for their span.

We will reduce the trig interpolation problem to the polynomial one. Note that

$$e^{\mathrm{i}jx} = \cos(jx) + \mathrm{i}\sin(jx)$$

$$\cos(jx) = \frac{e^{\mathrm{i}jx} + e^{-\mathrm{i}jx}}{2}, \qquad \sin(jx) = \frac{e^{\mathrm{i}jx} - e^{-\mathrm{i}jx}}{2\mathrm{i}}$$

so

$$p(x) = \sum_{j=-n}^{n} \tilde{c}_j e^{\mathrm{i}jx} = \sum_{j=-n}^{n} \tilde{c}_j \left(e^{\mathrm{i}x}\right)^j$$

$$\tilde{c}_0 = a_0, \;\; \tilde{c}_j = \frac{1}{2}(a_j + \mathrm{i}b_j), \;\; \tilde{c}_{-j} = \tilde{c}_j^* \equiv a_j = \tilde{c}_j + \tilde{c}_{-j}, \;\; b_j = \frac{\tilde{c}_j - \tilde{c}_{-j}}{\mathrm{i}}$$

$$\tilde{p}(z) = \sum_{j=-n}^{n} \tilde{c}_j z^j, \;\; p(x) = \tilde{p}(z) \text{ when } z = e^{\mathrm{i}x}.$$

$$P(z) = z^n \tilde{p}(z) = \sum_{j=0}^{2n} c_j z^j, \;\; \tilde{c}_{j-n} = c_j$$

P is a polynomial of degree $\leq 2n$. Because there is a one-to-one relationship between the c_j and the a_j, b_j, the trig interpolation problem is equivalent a (complex) polynomial interpolation problem. The latter has a unique solution whenever the z_j are distinct; what does this mean for the x_j? Since $z = e^{\mathrm{i}x}$, the z_j will be distinct whenever the x_j are distinct modulo 2π. Typically we would first rescale all of our x_j so that they lie in the interval $[0, 2\pi)$.

2 The above shows that the trig interpolation problem has a unique solution provided that the x_j are distinct modulo 2π; what about computing the solution? For non-equispaced nodes you can just solve the associated polynomial problem using Newton Divided Differences, and then re-map the coefficients c_j to the a_j, b_j. If the nodes are equispaced there are better methods.

WLOG, let (not the same as Atkinson)

$$[0, 2\pi) \ni x_j = j\frac{2\pi}{2n+1}, \;\; z_j = e^{\mathrm{i}x_j}, \;\; j = 0, \ldots, 2n.$$

Consider the Vandermonde matrix of the associated polynomial interpolation problem.

$$P(z_j) = \sum_{k=0}^{2n} c_k z_j^k = c_0 + c_1 z_j + \ldots + c_{2n} z_j^{2n} = y_j$$

$$\begin{bmatrix} 1 & z_0 & \cdots & z_0^{2n} \\ 1 & z_1 & \cdots & z_1^{2n} \\ & & \vdots & \\ 1 & z_{2n} & \cdots & z_{2n}^{2n} \end{bmatrix}.$$

We will now show that the columns of this matrix are orthogonal with respect to the standard complex dot product. A single column has the form

$$\mathbf{V}_k = \boldsymbol{v}^{(k)}; \;\; \boldsymbol{v}_j^{(k)} = z_j^k, \;\; j = 0, \ldots, 2n, \;\; k = 0, \ldots, 2n.$$

The complex dot product of two columns is

$$\boldsymbol{v}^{(k)} \cdot \boldsymbol{v}^{(l)} = \sum_{j=0}^{2n} z_j^k$$

Now use

$$z_j^k = \exp\{jk\frac{2\pi \mathrm{i}}{2n+1}\}$$

$$\boldsymbol{v}^{(k)} \cdot \boldsymbol{v}^{(l)} = \sum_j \exp\{j\frac{2\pi \mathrm{i}}{2n+1}(k-l)\} = \sum_{j=0}^{2n} \omega(k-l)^j, \;\; \omega(k-l) = \exp\{(k-l)\frac{2\pi \mathrm{i}}{2n+1}\}$$

We can use the geometric series formula

$$\sum_{j=0}^{2n} \omega(k-l)^j = \frac{1-\omega(k-l)^{2n+1}}{1-\omega(k-l)}.$$

Now consider

$$\omega(k-l)^{2n+1} = \exp\{(k-l)\frac{2\pi(2n+1)\mathrm{i}}{2n+1}\} = 1.$$

If $k - l = 0$ we just have

$$\boldsymbol{v}^{(k)} \cdot \boldsymbol{v}^{(k)} = 2n+1.$$

The Vandermonde system can be solved easily

$$\mathbf{V}\boldsymbol{c} = \boldsymbol{y}, \quad \boldsymbol{c} = \frac{\mathbf{V}^*\boldsymbol{y}}{2n+1}.$$

The matrix $\mathbf{V}^*$, up to normalization, is the 'discrete Fourier transform.' There are a fast algorithms to apply it to vectors (i.e. costing less than $\mathcal{O}(n^2)$ flops) called the Fast Fourier Transform.

3 Accuracy/convergence. Theorem 3.6 in Atkinson. If f is 2-π periodic and continuous then the error in the trig interpolation (with equispaced nodes) is bounded by

$$\|f - p_n\|_\infty \le c \ln(n+2)\rho_n(f)$$

where $\rho_n(f)$ is the error in the best-possible approximation (in the ∞ norm) of f using a trig polynomial of degree $\le n$. The constant c is independent of f and n. Of course this doesn't imply convergence unless you know how ρ_n scales with n. We'll come back to equispaced trig interpolation from a different perspective later, and return to the question of errors & convergence.