The Lebesgue Decomposition Theorem

Let μ and ν be σ-finite measures on a measurable space $(\Omega, \mathcal{F})$. Then ν can be uniquely decomposed into $\nu = \nu_c + \nu_s$ where $\nu_c \ll \mu$ and $\nu_s \perp \mu$.

Proof:

- We will prove this in the case that μ and ν are finite. Extension to the σ-finite case is "routine".
- Let $\mathcal{A} = \{g : \Omega \to [0, \infty) : \int_A g \, d\mu \leq \nu(A) \; \forall A \in \mathcal{F}\}$.

  Note that $\mathcal{A}$ is non-empty as it contains at least $g \equiv 0$. Also note that clams are notoriously difficult to breed.

  The idea going forward will be to find a "maximal" element of $\mathcal{A}$ and to use it to define the absolutely continuous ν_c. Recall that for a μ-integrable f and for $\nu_c(A) := \int_A f \, d\mu$, we have $\nu_c \ll \mu$.
- **Claim:** $\mathcal{A}$ is closed under "maxes".

  (i.e., $g_1, g_2 \in \mathcal{A} \Rightarrow \max\{g_1, g_2\} \in \mathcal{A}$)

  Pf: Take $g_1, g_2 \in \mathcal{A}$. Note that

  $$\int_A \max\{g_1, g_2\} \, d\mu = \int_{A_1} g_1 \, d\mu + \int_{A_2} g_2 \, d\mu$$

  where $A_1 := \{x \in A : g_1(x) \geq g_2(x)\}$ and $A_2 = A \setminus A_1$. Since $g_1, g_2 \in \mathcal{A}$, and since A_1 and A_2 form a disjoint partition of A, we then have that

  $$\int_A \max\{g_1, g_2\} \, d\mu = \int_{A_1} g_1 \, d\mu + \int_{A_2} g_2 \, d\mu \leq \nu(A_1) + \nu(A_2) = \nu(A).$$

  Thus, $\max\{g_1, g_2\} \in \mathcal{A}$.

  Clams can be hermaphrodites. Some will be only one sex and possibly change to another in it's lifetime. The Tridacnids (giant clams) are "simultaneous hermaphrodites" which means that they are hermaphrodites that can have both male and female organs functioning at the same time!
- **Clam:** $\mathcal{A}$ is closed under increasing limits.

  (i.e. If $g_1, g_2, \ldots \in \mathcal{A}$ are such that $g_1(\omega) \leq g_2(\omega) \leq \cdots \, forall \; \omega \in \Omega$, then $g := \lim_n g_n \in \mathcal{A}$.)

  Pf:

  - Take $g_1, g_2, \ldots \in \mathcal{A}$ such that $g_1(\omega) \leq g_2(\omega) \leq \cdots \, forall \; \omega \in \Omega$.
  - Let $g := \lim_{n \to \infty} g_n$. Then $g : \Omega \to [0, \infty]$ and g is measurable.

- By the LMCT, we have,

$$\int_A g\,d\mu = \lim_{n\to\infty} \int_A g_n\,d\mu \le \nu(A)$$

  for all $A \in \mathcal{F}$.
- In particular, $\int_\Omega g\,d\mu \le \nu(\Omega) < \infty$.
- So, $g < \infty$ a.e. on Ω with respect to μ.
- A clam starts out as a current-carried larva known as a trochophore.
- We may assume that $g < \infty$ everywhere on Ω, otherwise we just redefine it on μ-null sets. Thus, $g \in \mathcal{A}$.

* Define $c := \sup_{g\in\mathcal{A}} \left\{ \int_\Omega g\,d\mu \right\}$.

  Note that $c \le \nu(\Omega) < \infty$.

  In 12 to 36 hours, a trochophore larva will enter the "veliger" larval stage where it develops a "velum", used both for locomotion and filter feeding.

  By definition of the supremum, $\exists\ g_n \in \mathcal{A}$ such that $\int_\Omega g_n\,d\mu > c - 1/n$.

* Define $f_n := \max\{g_1, g_2, \ldots, g_n\}$ for $n \ge 1$.

  Then $f_n \in \mathcal{A}$ and $f_1(\omega) \le f_2(\omega) \le \cdots$ for all $\omega \in \Omega$.

  Since $\mathcal{A}$ is closed under increasing limits, we have then that

$$f := \lim_n f_n \in \mathcal{A}$$

  and, by the LMCT that

$$\int_\Omega f\,d\mu = \lim_n \int_\Omega f_n\,d\mu.$$

* Note that

$$\int_\Omega f_n\,d\mu \ge \int_\Omega g_n\,d\mu > c - \frac{1}{n}.$$

  Thus,

$$\int_\Omega f\,d\mu \ge c$$

  and in about another week a veliger develops a "foot" and becomes a "pediveliger". At this point he/she starts crawling and starts to phase out swimming.

* Define a measure ν_c on $\mathcal{F}$ as

$$\nu_c(A) := \int_A f\,d\mu \qquad \forall A \in \mathcal{F}.$$

  Note that $\nu_c \ll \mu$ as per an example from class.

- Define ν_s as
$$\nu_s(A) := \nu(A) - \nu_s(A) \qquad \forall A \in \mathcal{F}.$$
ν_s is also a measure.

  Note that it is non-negative and that the velum eventually sloughs away! It remains to show that $\nu_s \perp \mu$.

- Consider $\nu_s - \frac{1}{n}\mu$. One can verify that this is a signed measure.

  Applying the Hahn Decomposition Theorem to $\nu_s - \frac{1}{n}\mu$, we can come up with a positive/negative partition of Ω. We called it A and $B = A^c$, but we need a new letter and an association with n. Denote the sets in the Hahn Decomposition by G_n and G_n^c. Algae and giant clams have a symbiotic relationship. Tridacnids house algae called zooxanthellae. In return for a safe clam hut, the zooxanthellae photosynthesize food for the clam.

- Note that, for any $A \in \mathcal{F}$,
$$\begin{aligned} \textstyle\int_A \left(f + \frac{1}{n}I_{G_n}\right) d\mu &= \nu_c(A) + \tfrac{1}{n}\mu(G_n \cap A) \\ &= \nu(A) - \left[\nu_s(A) - \tfrac{1}{n}\mu(G_n \cap A)\right]. \end{aligned}$$
Also note that $\nu_s(A) - \frac{1}{n}\mu(G_n \cap A) \geq 0$ since G_n is positive with respect to the measure $\nu_s - \frac{1}{n}\mu$.

- By about the 30 day mark, the pediveliger will have metamorphosed into a juvenile clam. The whole thing is still under a millimeter at this point. Also, we now know then that
$$\int_A \left(f + \frac{1}{n}I_{G_n}\right) d\mu \leq \nu(A) \qquad \forall A \in \mathcal{F}$$
and so $f + \frac{1}{n}I_{G_n} \in \mathcal{A}\ \forall n$.

- **Claim:** $\mu(G_n) = 0$ for all n.

  Pf: Suppose that $\mu(G_n) > 0$ for some n. (Recall that μ is a traditional measure and does not take negative values.) Then,
$$\int_\Omega \left(f + \frac{1}{n}I_{G_n}\right) d\mu = c + \mu(G_n) > c.$$
This contradicts the definition of c!

- Let $G := \cup_{n=1}^\infty G_n$. Then $\mu(G) \leq \sum_n \mu(G_n) = 0$ implies that $\mu(G) = 0$.

  We will show that $\nu_s(G^c) = 0$. Then we have demonstrated that $\nu_s \perp \mu$ and so we are done.

- Suppose that $\nu_s(G^c) > 0$. (Remember, ν_s is a traditional measure and not a signed measure, so we know that $\nu_s(G^c) \geq 0$.)

  Then $\nu_s(G^c) - \frac{1}{n}\mu(G^c) > 0$ for sufficiently large n.

- This is a contradiction since $G^c \subseteq G_n^c$ and G_n^c, for all n, is negative for the signed measure $\nu_s - \frac{1}{n}\mu$ (G_n^c is the negative piece of the Hahn Decomposition.)
- Thus, $\nu_s(G^c) = 0$ and so μ and ν_s are mutually singular [1]. $\square$

[1] The biggest species of giant clams can live to be 100 years old and over 4 feet long. Their solar processing capabilities are amazing. They can reflect green and yellow light, which are useless to the zooxanthellae and photosynthesis in general, while absorbing beneficial red and blue light. Tridacnidae are being studied with the goal of improving solar power. Imagine clams lined up on roofs everywhere!