A Short Proof of the Unconditional Stability of the ADI-FDTD Scheme

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***Abstract* - The original proof for unconditional stability of the ADI-FDTD scheme for 3-D Maxwell's equations was algebraically complex, and could only be carried out with the help of significant use of computer-aided algebra. We present here a much simpler proof, which can be done by hand in a page or two. Since the new proof is based on an energy estimate (rather than a Fourier argument), it is more amenable to generalizations (for ex. to variable coefficients and/or to different types of boundary conditions).**

***Index Terms* - Alternating direction implicit (ADI) technique, FDTD method, stability, unconditional stability.**

I. INTRODUCTION

Finite difference time domain (FDTD) solutions of the 3-D Maxwell's equations are of increasing importance in a wide range of applications. The Yee scheme [1,2] has enjoyed much popularity due to its simple implementation and good flexibility. However, being an explicit scheme, it is subject to a CFL stability condition [2]. This condition becomes highly restrictive in cases where the resolution of the geometry forces the use of much finer space discretization than what is needed to solve for waves away from material irregularities or interfaces. The ADI-FDTD scheme, first presented by Zheng, Chen and Zhang [3], has received much attention thanks to its unconditional stability together with a similar accuracy and only slightly increased operation count (as compared to the Yee scheme). Namiki [4] demonstrates its practical advantages for two test problems (a monopole antenna near a thin dielectric wall, and a stripline with a narrow gap).

The purpose of this note is to demonstrate the ADI-FDTD unconditional stability in a much simpler way than how that result has previously been obtained [2,3]. Instead of directly showing that every Fourier mode has a growth factor of magnitude one (requiring very extensive algebra), we arrive here, after just a few lines of algebra, at an energy estimate that also rules out any Fourier mode growth factors larger than one. Unconditional stability (i.e. no CFL restriction for an intial value problem) then follows immediately (see e.g. [5] or [6] for definitions and for basic background material on stability conditions, alternative ways to demonstrate it, etc.)

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II. PROOF OF UNCONDITIONAL STABILITY

The ADI-FDTD scheme is normally laid out on a staggered Yee-type grid in space, but with no staggering in time. We can however just as well apply it on a regular grid in space (i.e. on a grid with every one of the 6 quantities E_x, E_y, E_z, H_x, H_y, H_z represented at each grid location, rather than at only one out of 8 such locations). This would be computationally uneconomical (by a factor of 8) since we would then effectively run 8 simultaneous uncoupled staggered computations. However, if we can prove unconditional stability in this regular grid case, we have also proven it for the 8 sub-problems, i.e. for the space-staggered ADI-FDTD scheme. For further notational simplicity, let us also assume that the space steps are equal: $\Delta x = \Delta y = \Delta z$, and we denote $\frac{\Delta t}{2\varepsilon\Delta x} = \alpha$, $\frac{\Delta t}{2\mu\Delta x} = \beta$ where ε is the medium permittivity and μ the permeability. Since we have a regular grid, we can use integer rather than a mix of integer and half-integer indices in space and time. We can now write the equations (8a-f) from [3], defining the ADI-FDTD algorithm, as follows (where we have separated the terms between the two sides according to their time level)

$$\frac{1}{\sqrt{\alpha}} E_x\big|_{i,j,k}^{n+1} - \sqrt{\alpha}\left(H_z\big|_{i,j+1,k}^{n+1} - H_z\big|_{i,j-1,k}^{n+1}\right) = \frac{1}{\sqrt{\alpha}} E_x\big|_{i,j,k}^{n} - \sqrt{\alpha}\left(H_y\big|_{i,j,k+1}^{n} - H_y\big|_{i,j,k-1}^{n}\right) \tag{1}$$

$$\frac{1}{\sqrt{\alpha}} E_y\big|_{i,j,k}^{n+1} - \sqrt{\alpha}\left(H_x\big|_{i,j,k+1}^{n+1} - H_x\big|_{i,j,k-1}^{n+1}\right) = \frac{1}{\sqrt{\alpha}} E_y\big|_{i,j,k}^{n} - \sqrt{\alpha}\left(H_z\big|_{i+1,j,k}^{n} - H_z\big|_{i-1,j,k}^{n}\right) \tag{2}$$

$$\frac{1}{\sqrt{\alpha}} E_z\big|_{i,j,k}^{n+1} - \sqrt{\alpha}\left(H_y\big|_{i+1,j,k}^{n+1} - H_y\big|_{i-1,j,k}^{n+1}\right) = \frac{1}{\sqrt{\alpha}} E_z\big|_{i,j,k}^{n} - \sqrt{\alpha}\left(H_x\big|_{i,j+1,k}^{n} - H_x\big|_{i,j-1,k}^{n}\right) \tag{3}$$

$$\frac{1}{\sqrt{\beta}} H_x\big|_{i,j,k}^{n+1} - \sqrt{\beta}\left(E_y\big|_{i,j,k+1}^{n+1} - E_y\big|_{i,j,k-1}^{n+1}\right) = \frac{1}{\sqrt{\beta}} H_x\big|_{i,j,k}^{n} - \sqrt{\beta}\left(E_z\big|_{i,j+1,k}^{n} - E_z\big|_{i,j-1,k}^{n}\right) \tag{4}$$

$$\frac{1}{\sqrt{\beta}} H_y\big|_{i,j,k}^{n+1} - \sqrt{\beta}\left(E_z\big|_{i+1,j,k}^{n+1} - E_z\big|_{i-1,j,k}^{n+1}\right) = \frac{1}{\sqrt{\beta}} H_y\big|_{i,j,k}^{n} - \sqrt{\beta}\left(E_x\big|_{i,j,k+1}^{n} - E_x\big|_{i,j,k-1}^{n}\right) \tag{5}$$

$$\frac{1}{\sqrt{\beta}} H_z\big|_{i,j,k}^{n+1} - \sqrt{\beta}\left(E_x\big|_{i,j+1,k}^{n+1} - E_x\big|_{i,j-1,k}^{n+1}\right) = \frac{1}{\sqrt{\beta}} H_z\big|_{i,j,k}^{n} - \sqrt{\beta}\left(E_y\big|_{i+1,j,k}^{n} - E_y\big|_{i-1,j,k}^{n}\right) \tag{6}$$

We next square of each side of each equation. For example, in the case of equations (1) and (5), this gives

$$\begin{aligned}\frac{1}{\alpha}\left(E_x\big|_{i,j,k}^{n+1}\right)^2 + \alpha\left(H_z\big|_{i,j+1,k}^{n+1} - H_z\big|_{i,j-1,k}^{n+1}\right)^2 - 2\left(E_x\big|_{i,j,k}^{n+1}\right)\left(H_z\big|_{i,j+1,k}^{n+1} - H_z\big|_{i,j-1,k}^{n+1}\right) = \\ = \frac{1}{\alpha}\left(E_x\big|_{i,j,k}^{n}\right)^2 + \alpha\left(H_y\big|_{i,j,k+1}^{n} - H_y\big|_{i,j,k-1}^{n}\right)^2 - 2\left(E_x\big|_{i,j,k}^{n}\right)\left(H_y\big|_{i,j,k+1}^{n} - H_y\big|_{i,j,k-1}^{n}\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{\beta}\left(H_y\big|_{i,j,k}^{n+1}\right)^2 + \beta\left(E_z\big|_{i,j+1,k}^{n+1} - E_z\big|_{i,j-1,k}^{n+1}\right)^2 - 2\left(H_y\big|_{i,j,k}^{n+1}\right)\left(E_z\big|_{i,j+1,k}^{n+1} - E_z\big|_{i,j-1,k}^{n+1}\right) = \\ = \frac{1}{\beta}\left(H_y\big|_{i,j,k}^{n}\right)^2 + \beta\left(E_x\big|_{i,j,k+1}^{n} - E_x\big|_{i,j,k-1}^{n}\right)^2 - 2\left(H_y\big|_{i,j,k}^{n}\right)\left(E_x\big|_{i,j,k+1}^{n} - E_x\big|_{i,j,k-1}^{n}\right)\end{aligned}$$

If we add these two equations, one of the expressions in the right hand side will become

$$-2\left(E_x\big|_{i,j,k}^{n}\right)\left(H_y\big|_{i,j,k+1}^{n}-H_y\big|_{i,j,k-1}^{n}\right)-2\left(H_y\big|_{i,j,k}^{n}\right)\left(E_x\big|_{i,j,k+1}^{n}-E_x\big|_{i,j,k-1}^{n}\right) \tag{7}$$

When summing this expression (7) over the full 3-D periodic volume, the sum cancels out to become zero (as can be seen directly, or by summation by parts; already summing in the z-direction makes it zero, and further summation in the x- and y-directions of zeros remain zero). In the same way, when we add the squares of all the relations (1) - (6) over the full volume, all the products that mix E and H-terms will cancel out on both of the time levels n and n-1. Hence, we get

$$\begin{aligned}
&\sum_{i,j,k}\Bigg\{\frac{1}{\alpha}\left\{(E_x|_{i,j,k}^{n+1})^2+(E_y|_{i,j,k}^{n+1})^2+(E_z|_{i,j,k}^{n+1})^2\right\}+\frac{1}{\beta}\left\{(H_x|_{i,j,k}^{n+1})^2+(H_y|_{i,j,k}^{n+1})^2+(H_z|_{i,j,k}^{n+1})^2\right\}+\\
&\quad+\alpha\left\{(H_x|_{i,j,k+1}^{n+1}-H_x|_{i,j,k-1}^{n+1})^2+(H_y|_{i+1,j,k}^{n+1}-H_y|_{i-1,j,k}^{n+1})^2+(H_z|_{i,j+1,k}^{n+1}-H_z|_{i,j-1,k}^{n+1})^2\right\}+\\
&\quad+\beta\left\{(E_x|_{i,j+1,k}^{n+1}-E_x|_{i,j-1,k}^{n+1})^2+(E_y|_{i,j,k+1}^{n+1}-E_y|_{i,j,k-1}^{n+1})^2+(E_z|_{i+1,j,k}^{n+1}-E_z|_{i-1,j,k}^{n+1})^2\right\}\Bigg\}=\\
&\sum_{i,j,k}\Bigg\{\frac{1}{\alpha}\left\{(E_x|_{i,j,k}^{n})^2+(E_y|_{i,j,k}^{n})^2+(E_z|_{i,j,k}^{n})^2\right\}+\frac{1}{\beta}\left\{(H_x|_{i,j,k}^{n})^2+(H_y|_{i,j,k}^{n})^2+(H_z|_{i,j,k}^{n})^2\right\}+\\
&\quad+\alpha\left\{(H_x|_{i,j+1,k}^{n}-H_x|_{i,j-1,k}^{n})^2+(H_y|_{i,j,k+1}^{n+1}-H_y|_{i,j,k-1}^{n})^2+(H_z|_{i+1,j,k}^{n}-H_z|_{i-1,j,k}^{n})^2\right\}+\\
&\quad+\beta\left\{(E_x|_{i,j,k+1}^{n}-E_x|_{i,j,k-1}^{n})^2+(E_y|_{i+1,j,k}^{n}-E_y|_{i-1,j,k}^{n})^2+(E_z|_{i,j+1,k}^{n}-E_z|_{i,j-1,k}^{n})^2\right\}\Bigg\}
\end{aligned} \tag{8}$$

We denote the two sums in (8) as $\Sigma_{(1)}^{n+1}$ and $\Sigma_{(2)}^{n}$ respectively. The superscript denotes the time level, and the subscript indicates that the two sums furthermore differ subtly in the indices for the terms that contain differences.

Exactly in the same way as we have arrived at $\Sigma_{(1)}^{n+1}=\Sigma_{(2)}^{n}$ starting from (8a-f) in [3], starting instead from (9a-f) in [3] gives $\Sigma_{(2)}^{n+2}=\Sigma_{(1)}^{n+1}$. This tells that $\Sigma_{(2)}^{n+2}=\Sigma_{(2)}^{n}$, i.e. after any even number of (partial) time steps, $\Sigma_{(2)}$ repeats identically (and similarly, $\Sigma_{(1)}$ repeats after any odd number of (partial) time steps). The expression

$$\sum_{i,j,k}\left\{\frac{1}{\alpha}\left\{(E_x|_{i,j,k}^{n})^2+(E_y|_{i,j,k}^{n})^2+(E_z|_{i,j,k}^{n})^2\right\}+\frac{1}{\beta}\left\{(H_x|_{i,j,k}^{n})^2+(H_y|_{i,j,k}^{n})^2+(H_z|_{i,j,k}^{n})^2\right\}\right\} \tag{9}$$

is common to both $\Sigma_{(1)}^{n}$ and $\Sigma_{(2)}^{n}$. The additional terms in these two sums are all perfect squares and therefore guaranteed to be positive. Hence, the sum (9) is bounded for all times by the initial values of $\Sigma_{(1)}$ and $\Sigma_{(2)}$. The ADI-FDTD scheme can therefore not possess any Fourier mode with exponential growth and - just like from the previous all-Fourier based proof - unconditional stability is established.

Both the previous and the present proofs can immediately be extended to the case of higher order finite difference approximations in space. Another possible direction of generalization - which only the present proof would appear to be amenable to - is to variable coefficients and to different boundary conditions. Although we have not carried out such extensions here, they are usually possible in case of energy-type bounds of the present type. The reader is referred to Richtmyer and Morton [5] and Gustafsson et.al. [7] for further information.

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