Applied Math 5610 **Homework 10** **Due: April 17, 2019**
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1. a. Determine the stability domain for the *leap-frog* scheme

$$y_{n+1} - y_{n-1} = 2k\, f(t_n, y_n)$$

for solving the ODE $y' = f(t, y)$ (with k denoting the time step: $k = t_{i+1} - t_i$).

b. Determine the leap-frog scheme's order of accuracy.

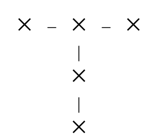
c. Consider next the following variation of the leap-frog scheme:

$$y_{n+1} - y_{n-1} = k\left(\tfrac{7}{3} f(t_n, y_n) - \tfrac{2}{3} f(t_{n-1}, y_{n-1}) + \tfrac{1}{3} f(t_{n-2}, y_{n-2})\right) .$$

It can be shown that this modified right hand side (RHS) has made the scheme third order accurate. Determine its stability domain.

d. Can this modified leap-frog scheme be used to solve $y' = y$ and/or $y' = -y$? Explain!

2. Consider for the heat equation $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$, $\sigma > 0$, the scheme with the stencil shape



shown to the right and which is second order in both time and space.

a. Explain, based on general ODE principles, why this scheme will be unconditionally stable in the limit of $h, k \to 0$.

b. Carry out von Neumann stability analysis to reach the same conclusion.

3. A very straightforward attempt to numerically solve $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0$ would be to use Forward Euler in time and centered, second order finite differences in space:

$$\frac{u(x,t+k) - u(x,t)}{k} + \frac{-\frac{1}{2}u(x-2h,t) + u(x-h,t) - u(x+h,t) + \frac{1}{2}u(x+2h,t)}{h^3} = 0 \qquad (1)$$

a. Explain, based on some general principle, why (1) is unconditionally unstable.

One attempt to get around the instability in (1) would be too try a Lax-Friedrich – type approximation

$$\frac{u(x,t+k)-\frac{1}{2}(u(x-h,t)+u(x+h,t))}{k}+\frac{-\frac{1}{2}u(x-2h,t)+u(x-h,t)-u(x+h,t)+\frac{1}{2}u(x+2h,t)}{h^3}=0 \qquad (2)$$

b. Show that (2) is *consistent* only if $k/h^2 \to \infty$ as $h,k \to 0$.

c. Show that (2) is *stable* only if $k/h^3 < \frac{1}{4}$ as $h,k \to 0$.

Hence, from the results of (b) and (c), we can conclude that (2), in spite of being conditionally stable, cannot be used to solve the PDE.

4. Consider the PDE $\frac{\partial u}{\partial t}+i\frac{\partial^2 u}{\partial x^2}=0$ (where $i=\sqrt{-1}$) . We wish to approximate the Cauchy problem using standard second order FD in space, and in time using either

(i) Leap-frog (centered difference),
(ii) Forward Euler,
(iii) Crank-Nicholson (AM2)

Determine in each of these cases the stability restriction (if any) on *k* and *h* (time and space step).