Applied Math 5610 **Homework 9** **Due: April 10, 2019**
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1. a. Consider the PDE $\frac{\partial u}{\partial t} = (a+ib)\frac{\partial u}{\partial x}$ where $u = u(x,t)$ and a, b are real constants, Show this is ill posed as a Cauchy problem (initial value problem on $x \in [-\infty, \infty]$, t increasing), if and only if $b \neq 0$.

For parts *b* and *c*, consider the system of PDEs $\frac{\partial}{\partial t}\begin{bmatrix} \underline{u} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} \underline{u} \end{bmatrix}$, where A is a constant matrix.

b. Show that this system is well posed if A is a Hermitian matrix (satisfying $A^* = A$).

c. Show that the system is ill posed if A has a non-real eigenvalue.

2. Normally, a variable coefficient problem is well posed if and only if all locally 'frozen coefficient' problems are well posed. A famous (or infamous) failure of this rule is offered by

$$\frac{\partial u}{\partial t} = (i\cos x)\frac{\partial u}{\partial x} + (i\sin x)\frac{\partial^2 u}{\partial x^2} \qquad \left[= i\frac{\partial}{\partial x}\left(\sin x \frac{\partial u}{\partial x}\right)\right].$$

We know from Problem 1 a above that, with the variable coefficients 'frozen' at their values at $x = 0$, the resulting equation is ill posed. Show that nevertheless, the variable coefficient PDE is well posed by means of demonstrating that $\frac{d}{dt}\|u\|^2 = 0$.

<u>Note:</u> We define $\|u\|^2 = \int_{-\infty}^{\infty} \bar{u}\, u\, dx$, and assume that u and its derivatives decay to zero in both directions.

3. a. The Lecture notes on the class web page "Forward Euler for heat equation - stability condition" show the result of time stepping the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ using Forward Euler (FE) in time and FD2 (second order centered finite differences) in space, with initial condition (IC)

$$u(x,0) = \begin{cases} 2x & \text{if} \quad 0 \le x \le \frac{1}{2} \\ 2-2x & \text{if} \quad \frac{1}{2} \le x \le 1 \end{cases} .$$

Reproduce the results that were shown in these notes.

b. Replace the Forward Euler time stepper with your own implementation of RK4 (the standard fourth order Runge-Kutta scheme). This ODE solver's stability domain includes the negative real axis down to $\xi \approx -2.78529$. Run the same problem as above, but with this RK4 solver, and verify that using time steps $k = \Delta t$ just below and just above the critical one indeed gives smooth solutions vs. jagged blow-up.

4. The RK4 stability domain is described by the equation $r = 1 + \xi + \frac{1}{2}\xi^2 + \frac{1}{6}\xi^3 + \frac{1}{24}\xi^4$. To determine its extent along the imaginary axis, let $\xi = i\alpha$, with α real, and then solve the equation $|r(\alpha)|^2 = 1$ to arrive at the solutions $\alpha = \pm 2\sqrt{2}$.