1. a. Consider the PDE \( \frac{\partial u}{\partial t} + i (a + ib) \frac{\partial u}{\partial x} \) where \( u = u(x,t) \) and \( a, b \) are real constants. Show this is ill posed as a Cauchy problem (initial value problem on \( x \in [-\infty, \infty] \), \( t \) increasing), if and only if \( b \neq 0 \).

For parts b and c, consider the system of PDEs \( \frac{\partial}{\partial t} \begin{bmatrix} u \\ \frac{\partial u}{\partial x} \end{bmatrix} = A \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial x} \end{bmatrix} \), where \( A \) is a constant matrix.

b. Show that this system is well posed if \( A \) is a Hermitian matrix (satisfying \( A^* = A \)).

c. Show that the system is ill posed if \( A \) has a non-real eigenvalue.

2. Normally, a variable coefficient problem is well posed if and only if all locally ‘frozen coefficient’ problems are well posed. A famous (or infamous) failure of this rule is offered by

\[
\frac{\partial u}{\partial t} = (i \cos x) \frac{\partial u}{\partial x} + (i \sin x) \frac{\partial^2 u}{\partial x^2} = i \frac{\partial}{\partial x} \left( \sin x \frac{\partial u}{\partial x} \right).
\]

We know from Problem 1 a above that, with the variable coefficients ‘frozen’ at their values at \( x = 0 \), the resulting equation is ill posed. Show that nevertheless, the variable coefficient PDE is well posed by means of demonstrating that \( \frac{d}{dt} \| u \|^2 = 0 \).

Note: We define \( \| u \|^2 = \int_{-\infty}^{\infty} u^* u \, dx \), and assume that \( u \) and its derivatives decay to zero in both directions.

3. a. The Lecture notes on the class web page “Forward Euler for heat equation - stability condition” show the result of time stepping the heat equation \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \) using Forward Euler (FE) in time and FD2 (second order centered finite differences) in space, with initial condition (IC)

\[
u(x,0) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}.
\]

Reproduce the results that were shown in these notes.

b. Replace the Forward Euler time stepper with your own implementation of RK4 (the standard fourth order Runge-Kutta scheme). This ODE solver’s stability domain includes the negative real axis down to \( \xi \approx -2.78529 \). Run the same problem as above, but with this RK4 solver, and verify that using time steps \( k = \Delta t \) just below and just above the critical one indeed gives smooth solutions vs. jagged blow-up.

4. The RK4 stability domain is described by the equation \( r = 1 + \xi + \frac{1}{2} \xi^2 + \frac{1}{3} \xi^3 + \frac{1}{4} \xi^4 \). To determine its extent along the imaginary axis, let \( \xi = i \alpha \), with \( \alpha \) real, and then solve the equation \( |r(\alpha)|^2 = 1 \) to arrive at the solutions \( \alpha = \pm 2 \sqrt{2} \).