Applied Math 5610 **Homework 7** **Due: March 18, 2019**
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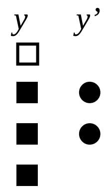
1. Consider linear multi-step schemes of the form 

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a. Give coefficients in one scheme that fits within this form, is *second* order accurate, and satisfies the root condition.

b. Demonstrate that no scheme of this form can be *third* order accurate and still satisfy the root condition.

Note: Dahlquist's first stability barrier does not rule out such a scheme – it just happens not to be possible anyway.

Hint: We can first simplify by setting $h=1$, and then let the scheme be $y_{n+1}+a_0 y_n+a_1 y_{n-1}+a_2 y_{n-2}+b_0 y'_n+b_1 y'_{n-1}=0$. The third order accuracy requirement gives 4 linear equations between the 5 coefficients. If we for example let b_1 be a 'free' parameter, show that the other coefficients can be expressed in terms of b_1 as

$$\begin{cases} a_0 &= \frac{1}{4}(6-5b_1) \\ a_1 &= -3+b_1 \\ a_2 &= \frac{1}{4}(2+b_1) \\ b_0 &= \frac{1}{4}(-6+b_1) \end{cases}$$

At this point, the easiest procedure is probably to just plot the maximum magnitude of the roots that need to be tested for the root condition, as function of b_1, and from this obtain that no choice for b_1 is satisfactory.

c. Generate the coefficients in a *fourth* order accurate scheme of the specified form.

2. The following are six suggestions for linear multistep schemes for solving $y' = f(x,y)$:

a. $y_{n+1}=\frac{1}{2}y_n+\frac{1}{2}y_{n-1}+2hf_n$

b. $y_{n+1}=y_n+2hf_{n-1}$

c. $y_{n+1}=y_{n-3}+4hf_n$

d. $y_{n+1}=-\frac{27}{11}(y_n-y_{n-1})+y_{n-2}+\frac{3}{11}h(f_{n+1}+9f_n+9f_{n-1}+f_{n-2})$

e. $y_{n+1}=\frac{8}{19}(y_n-y_{n-2})+y_{n-3}+\frac{6}{19}h(f_{n+1}+4f_n+4f_{n-2}+f_{n-3})$

f. $y_{n+1}=3y_n-2y_{n-1}+h(\frac{13}{12}f_{n+1}-\frac{5}{3}f_n-\frac{5}{12}f_{n-1})$

The incomplete table below summarizes key properties of these schemes. Complete the missing table entries (you do not need to supply any derivations).

case	characteristic equation	roots	stabi-lity	accu-racy	consis-tency	leading error term	convergence to solution
a	$r^2 - \frac{1}{2}r - \frac{1}{2} = 0$	$1, -\frac{1}{2}$	Yes	0		$-\frac{1}{2}hf'(\xi)$	
b					No		No
c					Yes		
d						$-\frac{3}{1540}h^7 f^{(7)}(\xi)$	No
e	$r^4 - \frac{8}{19}r^3 + \frac{8}{19}r - 1 = 0$	$\pm 1, \frac{4}{19} \pm \frac{\sqrt{345}}{19}i$		6		$-\frac{6}{665}h^7 f^{(7)}(\xi)$	
f				2	Yes		

3. The following Matlab code produces the figure shown to the right:

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r = exp(complex(0,linspace(0,2*pi)));
xi = (11/6*r.^3-3*r.^2+3/2*r-1/3)./r.^3;
plot(xi,'LineWidth',2)
axis equal; ax = gca; box off;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
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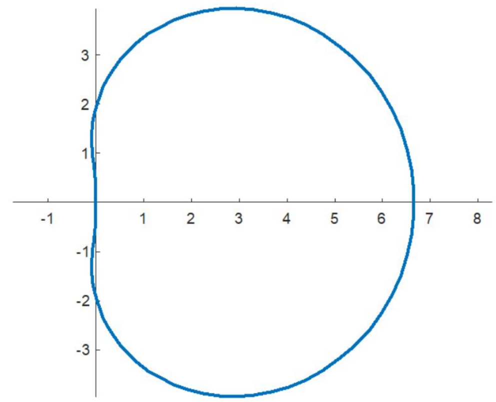

This figure displays the boundary of the stability domain for a certain linear multistep method (LMM).

a. Write down the formula for this LMM in the conventional form of coefficients for $y(x)$ and $y'(x)$ at equispaced x-values. Does this scheme go under a well-known name?

b. Determine if the stability domain is given by the inside or the outside (or neither) of the shown curve.

c. Determine if the scheme satisfies the root condition.

4. The following is an example of a predictor-corrector scheme for solving $y' = f(x, y)$ and which is not derived from the Adams-Bashforth / Adams-Moulton formulas:

$$y^*_{n+1} = y_{n-1} + 2hf_n$$

$$y_{n+1} = \tfrac{4}{5}y_n + \tfrac{1}{5}y_{n-1} + \tfrac{2}{5}h(f(x_{n+1}, y^*_{n+1}) + 2f(x_n, y_n))$$

a. Determine the order of accuracy of the scheme.

b. Check the root condition for stability.

c. The figure to the right shows the stability domain for the method. Verify that it extends along the imaginary axis and the negative real axis to exactly $\pm 1.5i$ and -1.5, respectively.