Applied Math 5610 **Homework 3** **Due: February 6, 2019**
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1. Each step of the symmetric QR algorithm transforms a symmetric, tridiagonal matrix $A_1 = Q_1 R_1$ into $R_1 Q_1 = A_2$ (again symmetric and tridiagonal). We denote the elements as follows (with all numbers assumed to be real):

$$A_1 = \begin{bmatrix} a_1 & b_2 & & & \\ b_2 & a_2 & b_3 & & \\ & b_3 & a_3 & b_4 & \\ & & \ddots & \ddots & \ddots \\ & & & b_n & a_n \end{bmatrix}, \quad A_2 = \begin{bmatrix} \bar{a}_1 & \bar{b}_2 & & & \\ \bar{b}_2 & \bar{a}_2 & \bar{b}_3 & & \\ & \bar{b}_3 & \bar{a}_3 & \bar{b}_4 & \\ & & \ddots & \ddots & \ddots \\ & & & \bar{b}_n & \bar{a}_n \end{bmatrix}.$$

It turns out that all the steps involved in going from A_1 to A_2 can be formulated conveniently as simple recursions. The 'classic' book *The Algebraic Eigenvalue Problem* by J.H. Wilkinson (from 1965) gives these recursions as follows:

$$u_0 = 0, \quad c_0 = 1, \quad b_{n+1} = 0, \quad a_{n+1} = 0,$$

$$\left.\begin{aligned} \gamma_i &= a_i - u_{i-1} & & \\ p_i^2 &= \gamma_i^2 / c_{i-1}^2 & & (\text{if } c_{i-1} \neq 0) \\ &= c_{i-2}^2 b_i^2 & & (\text{if } c_{i-1} = 0) \\ \bar{b}_i^2 &= s_{i-1}^2 (p_i^2 + b_{i+1}^2) & & (\text{if } i \neq 1) \\ s_i^2 &= b_{i+1}^2 / (p_i^2 + b_{i+1}^2) & & \\ c_i^2 &= p_i^2 / (p_i^2 + b_{i+1}^2) & & \\ u_i &= s_i^2 (\gamma_i + a_{i+1}) & & \\ \bar{a}_i &= \gamma_i + u_i & & \end{aligned}\right\} \quad (i = 1, 2, \ldots, n)$$

The algorithm above calculates $\bar{a}_i$ and $\bar{b}_i^2$. This suffices since the signs of the $\bar{b}_i$ have no influence on the eigenvalues.

a. Verify that A_1 being symmetric and tridiagonal implies that A_2 will also become of this same form.

b. Show that the eigenvalues of A_2 indeed are unchanged, no matter what signs are given to the $\bar{b}_i$ entries.

c. Implement the algorithm above in Matlab, and reproduce the results in Atkinson, page 624.

d. Furthermore, incorporate single shifts, and then compare with the example in Atkinson, page 628.

For parts c and d, note in particular how the element $\bar{b}_3$ decays during successive iterations. Describe the convergence rate that you see. Try to make your code effective with regard to both computer time and memory. In particular, note that no arrays are needed beyond the two which hold the a_i and b_i^2.

Note: It is possible for shifted QR to get 'stuck'. If that happens, inserting a single random shift usually resolves this issue. Such a shift can for example be done during the very first iteration. However, no such extra shift is needed for solving the present problems.

2. In the Lecture notes "Calculation of eigenvectors" on the class web page, the Newton iteration for solving the nonlinear system

$$\begin{cases} A\underline{x} & = & \lambda \underline{x} \\ \underline{x}^* \underline{x} & = & 1 \end{cases}$$

is stated to take the form

$$\left[\begin{array}{c|c} A-\lambda_i I & \underline{x}_i \\ \hline \underline{x}_i^* & 0 \end{array}\right] \left[\begin{array}{c} \Delta\underline{x}_i \\ \hline -\Delta\lambda_i \end{array}\right] = \left[\begin{array}{c} -\underline{r}_i \\ \hline -\underline{p}_i \end{array}\right], \quad i = 0,1,2,3,\ldots$$

where $\begin{cases} \underline{r}_i & = & A\underline{x}_i - \lambda_i \underline{x}_i \\ p_i & = & \frac{1}{2}(\underline{x}^* \underline{x} - 1) \end{cases}$.

a. Verify this assertion.

b. Let A be the Hilbert matrix in Atkinson, equation (9.1.12), page 593 (obtained in Matlab by the statement `A = hilb(3);`). Choose some random initial guess for both λ_0 and $\underline{x}_0$, and run the iteration above until it has converged (to double precision 10^{-16}). Describe the convergence rate that you observe for the λ-iterates towards one of the eigenvalues that Atkinson lists in equation (9.1.15).

3. Let A be a $n \times n$ matrix, and let $\underline{a}_j$ be its j^{th} column. Prove Hadamard's inequality $|\det A| \le \prod_{j=1}^{n} \|\underline{a}_j\|_2$.

Hint: Consider a QR decomposition of A.

4. Let A be a real $n \times n$ matrix with non-negative elements, satisfying $\sum_{j=1}^{n} a_{ij} = 1,\ i = 1,2,\ldots,n.$ Determine the spectral radius $\rho(A)$.