Applied Math 5610 **Homework 2** **Due: January 30, 2019**
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1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

   a. Based on Gershgorin's theorem, deduce that $O(\varepsilon)$ perturbations of the matrix elements can at most move the eigenvalues by $O(\varepsilon^{1/3})$ (where we assume that ε is a small quantity).

   b. Give an example of an $O(\varepsilon)$ perturbation for which you can show that the eigenvalues indeed have moved by as much as $O(\varepsilon^{1/3})$.

2. Given any two vectors $\underline{x}$ and $\underline{y}$ satisfying (i) $\underline{x}^* \underline{x} = \underline{y}^* \underline{y}$ and (ii) $\underline{x}^* \underline{y}$ is real, and select $\omega = \dfrac{\underline{y} - \underline{x}}{\| \underline{y} - \underline{x} \|}$ when forming a Householder matrix *H*. Verify by direct evaluation that $H\,\underline{x} = (I - 2\underline{\omega}\,\underline{\omega}^*)\underline{x}$ indeed becomes equal to $\underline{y}$.

3. Let H be a Householder matrix (of size $n \times n$), i.e. $H = I - 2\underline{\omega}\,\underline{\omega}^*$ where $\underline{\omega}^* \underline{\omega} = 1$.

   a. Noting that *H* is both unitary and Hermitian, show that its only possible eigenvalues are +1 and -1.

   b. By using the result in (a) and by noting that the Trace of H (Tr (*H*)) is equal to $n-2$ (show this!), determine all the eigenvalues of *H*.

   c. Show that $H\underline{\omega} = -\underline{\omega}$ and that $H\underline{v} = \underline{v}$ if $\underline{v}^* \underline{\omega} = 0$.

   d. Determine (again) all the eigenvalues of *H* by using that the result in part (c) provided all the eigenvalues.

   e. Use the knowledge of the eigenvalues of *H* to determine det (*H*).

   f. Determine det (*H*) by 'brute force', i.e. write out the matrix and show how its value can be obtained by the usual rules for determinants (i.e. adding multiples of rows to each other, expanding along rows/columns, etc.)

4. a. Let *A* be a symmetric, irreducible tridiagonal matrix (i.e. there are no zero elements on the two diagonals next to the main one). Show that *A* cannot have a multiple eigenvalue.

   b. Let *A* be an upper Hessenberg matrix with all its sub-diagonal elements non-zero. Assume *A* has a multiple eigenvalue. Show that there can only be one eigenvector associated with it.

   Hints: For both parts (a) and (b): Consider the diagonal or Jordan canonical form of $A - \lambda I$ and its rank. An upper Hessenberg (UH) matrix is zero in all entries below its first sub-diagonal.