APPM 4/5560

Problem Set Nine (Due Wednesday, April 17th)

1. Let $\{N(t)\}$ be a Poisson process with rate λ.
   (a) Write down the generator matrix for $\{N(t)\}$.
   (b) Write down a closed form expression for $p_{ij}(t)$.
   (c) Write out the Kolmogorov forward equation for the process and verify that your answer to part (b) is a solutions.

2. [**Simulation**] Simulate arrival times for a Poisson process with rate $\lambda = 2.5$. Count the number of arrivals you observe up to time $T = 10$. Repeat $100,000$ times. From the $100,000$ values obtained, estimate the distribution of the number of arrivals in $[0, T]$ and compare to the true distribution. Hand in your code.

3. Potential customers arrive at a single-server station in accordance with a Poisson process with rate λ. If the arriving customers finds n customers already in the station, then he/she will enter the system with probability α_n. Otherwise, he/she will leave. Assuming an exponential service time with rate μ, set this up as a birth and death process and determine the birth and death rates. (You do not have to do the formal "little oh approach".)

4. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean $1/4$ hour.
   (a) What is the average number of customers in the shop?
   (b) If the barber could work twice as fast, how much more business would he do?

5. Imagine a 3×4 lattice with 12 positions, 5 of which are occupied by a single particle. Each particle will jump to a new unoccupied site with rate 1. The site is chosen at random from the 7 unoccupied sites. Find the stationary distribution for the set of occupied sites. (Note: Label the sites as $1, 2, \ldots, 12$. The answer will be some distribution $\pi(i_1, i_2, i_3, i_4, i_5)$ where i_1, i_2, i_3, i_4, i_5 are distinct values in $\{1, 2, \ldots, 12\}$.)

6. [**Required for 5560 Students Only**] (Durrett 4.24) *Kolmogorov Cycle Condition.* Consider an irreducible Markov chain with state space S. We say that the "cycle condition" is satisfied if , given a cycle of states $x_0, x_1, \ldots, x_n = x_0$, with positive transition rates ($q_{x_{i-1},x_i} > 0$) for $i = 1, 2, \ldots, n$, we have
$$\prod_{i=1}^{n} q_{x_{i-1},x_i} = \prod_{i=1}^{n} q_{x_i,x_{i-1}}.$$
   (a) Show that if q has a stationary distribution π that satisfies the detailed balance condition, the cycle condition must hold.
   (b) The converse: Suppose that the cycle condition holds. Let $a \in S$ and set $\pi_a = c$. For $b \neq a$, $b \in S$, let $x_0 = a, x_2, \ldots, x_k = b$ be a path from a to b with $q_{x_{i-1},x_i} > 0$ for $i = 1, 2, \ldots, k$. Define
$$\pi_b = \prod_{i=1}^{k} \frac{q_{x_{i-1},x_i}}{q_{x_i,x_{i-1}}}.$$
   Show that π_b is well-defined, i.e., is independent of the path chosen. Then conclude that π satisfies the detailed balance condition.