1. Let \( \{N(t)\} \) be a Poisson process with rate \( \lambda \).
   (a) Write down the generator matrix for \( \{N(t)\} \).
   (b) Write down a closed form expression for \( p_{ij}(t) \).
   (c) Write out the Kolmogorov forward equation for the process and verify that your answer to part (b) is a solution.

2. [Simulation] Simulate arrival times for a Poisson process with rate \( \lambda = 2.5 \). Count the number of arrivals you observe up to time \( T = 10 \). Repeat 100,000 times. From the 100,000 values obtained, estimate the distribution of the number of arrivals in \([0, T]\) and compare to the true distribution. Hand in your code.

3. Potential customers arrive at a single-server station in accordance with a Poisson process with rate \( \lambda \). If the arriving customers finds \( n \) customers already in the station, then he/she will enter the system with probability \( \alpha_n \). Otherwise, he/she will leave. Assuming an exponential service time with rate \( \mu \), set this up as a birth and death process and determine the birth and death rates. (You do not have to do the formal “little oh approach”.)

4. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean \( 1/4 \) hour.
   (a) What is the average number of customers in the shop?
   (b) If the barber could work twice as fast, how much more business would he do?

5. Imagine a \( 3 \times 4 \) lattice with 12 positions, 5 of which are occupied by a single particle. Each particle will jump to a new unoccupied site with rate 1. The site is chosen at random from the 7 unoccupied sites. Find the stationary distribution for the set of occupied sites. (Note: Label the sites as 1, 2, \ldots, 12. The answer will be some distribution \( \pi(i_1, i_2, i_3, i_4, i_5) \) where \( i_1, i_2, i_3, i_4, i_5 \) are distinct values in \( \{1, 2, \ldots, 12\} \).)

6. [Required for 5560 Students Only] (Durrett 4.24) Kolmogorov Cycle Condition. Consider an irreducible Markov chain with state space \( S \). We say that the “cycle condition” is satisfied if, given a cycle of states \( x_0, x_1, \ldots, x_n = x_0 \), with positive transition rates \( (q_{x_{i-1},x_i} > 0) \) for \( i = 1, 2, \ldots, n \), we have
   \[
   \prod_{i=1}^{n} q_{x_{i-1},x_i} = \prod_{i=1}^{n} q_{x_i,x_{i-1}}.
   \]
   (a) Show that if \( q \) has a stationary distribution \( \pi \) that satisfies the detailed balance condition, the cycle condition must hold.
   (b) The converse: Suppose that the cycle condition holds. Let \( a \in S \) and set \( \pi_a = c \). For \( b \neq a, b \in S \), let \( x_0 = a, x_2, \ldots, x_k = b \) be a path from \( a \) to \( b \) with \( q_{x_{i-1},x_i} > 0 \) for \( i = 1, 2, \ldots, k \). Define
   \[
   \pi_b = \prod_{i=1}^{k} q_{x_{i-1},x_i}.
   \]
   Show that \( \pi_b \) is well-defined, i.e., is independent of the path chosen. Then conclude that \( \pi \) satisfies the detailed balance condition.