APPM 5600: Homework #9
Due in class Monday December 11

1 We saw in class that you can compute quadrature weights in 2 ways: (i) $w_i = \int_a^b \ell_i(x)\mathrm{d}x$ and (ii) solve a transposed Vandermonde system. Prove that these methods give you the same answer. Hint: Let $\mathbf{V}$ be the Vandermonde matrix and recall that the columns of $\mathbf{V}^{-1}$ are the coefficients (in the monomial basis) of the Lagrange interpolating polynomials.

2 Some nodes free, others fixed

(a) (Gauss-Radau) Let $x_0 = -1$. Find nodes x_1, x_2 and x_3 and weights $w_0, \ldots, w_3$ such that the quadrature

$$\sum_{i=0}^{3} w_i f(x_i) \approx \int_{-1}^{1} f(x)\mathrm{d}x$$

integrates polynomials of as high degree as possible exactly.

(b) (Gauss-Lobatto) Let $x_0 = -1$ and $x_3 = 1$. Find nodes x_1 and x_2 and weights $w_0, \ldots, w_3$ such that the quadrature

$$\sum_{i=0}^{3} w_i f(x_i) \approx \int_{-1}^{1} f(x)\mathrm{d}x$$

integrates polynomials of as high degree as possible exactly.

3 Use the Peano kernel to show that the corrected trapezoid rule

$$\int_0^h f(x)\mathrm{d}x \approx \frac{h}{2}[f(h) + f(0)] + \frac{h^2}{12}[f'(0) - f'(h)]$$

has error bounded by

$$\frac{h^5}{720}\|f^{(4)}\|_\infty.$$

4 The Peano kernel error formula plus the integral mean value theorem implies that the error in Simpson's rule (not composite) is

$$\text{error } = f^{(4)}(\xi)\int_{-h}^{h} K(t)\mathrm{d}t = f^{(4)}(\xi)\frac{h^5}{90}$$

for some $\xi \in [-h, h]$.

(a) Show that the composite Simpson's rule has an asymptotic error formula of the form

$$\lim_{n\to\infty} \frac{\text{error}}{h^4} = \frac{1}{180}[f^{(3)}(b) - f^{(3)}(a)].$$

(b) Compute the Simpson's rule approximation to $\int_{-1}^{1}(1 + x^2)^{-1}\mathrm{d}x$ using $n = 128$ (129 points including endpoints). Then, without computing any more function values, use Aitken extrapolation to produce an improved estimate. Compare the two estimates to the true value.