APPM 4/5560

Problem Set Eight (Due Wednesday, April 3rd)

1. Consider a machine that is either in an operating state or a repair state. Suppose that, when it is operating, it stays that way for an exponential amount of time with rate λ and then, when in for repair, the repair takes an exponential amount of time with rate μ. Further suppose that all of these exponentials are independent.

   Let
   $$X(t) = \begin{cases} 0 & , \quad \text{if the machine is operating at time } t \\ \\ 1 & , \quad \text{if the machine is in repair at time } t \end{cases}$$

   Show that $\{X(t)\}$ is a birth and death process and give the birth and death rates.

2. Each individual in a biological population is assumed to give birth after an exponential amount of time with rate λ, and to die after an exponential amount of time with rate μ. In addition, new individuals are immigrating in to the population according to a Poisson process with rate θ, however, immigration is not allowed when the population size is N or larger.

   Let $X(t)$ be the number of individuals in the population at time t. Show that $\{X(t)\}$ is a birth and death process and give the birth and death rates.

   [Hint: For any one individual, the probability it gives birth in any interval $(t, t+h]$ is $P(B \leq h)$ where $B \sim exp(rate = \lambda)$. This is due to the lack of memory property of the exponential. Show that, for small h, this probability is $\lambda h + o(h)$. Now this is like other problems we have done in class where the word "exponential" was not mentioned, but instead we said something like "the individual birth rate is λ". Note that the meaning of the phrase "the individual birth rate is λ" was that the probability an individual gives birth in a small interval of length h is $\lambda h + o(h)$. The same sort of thing can be said about the individual exponential death times. Okay, so now this hint is longer than the problem so that's kind of weird.]

3. Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda$, $i \geq 0$, and death rates $\mu_i = i\mu$, $i \geq 0$.

   (a) Determine the expected time to go from state 0 to state 4.

   (b) Determine the expected time to go from state 2 to state 5.

4. Each time a machine is repaired, it remains up and working for an exponentially distributed time with rate λ. It then fails, and its failure is either of two types. If it is a type 1 failure, then the time to repair the machine is exponentially distributed with mean μ_1; if it is a type 1 failure, then the time to repair the machine is exponentially distributed with mean μ_2. Each failure is, independently of the time it took the machine to fail, a type 1 failure with probability p and a type 2 failure with probability $1-p$.

   Write down the generator matrix for this birth and death process.

5. **[Required for 5560 only]** Let $\{X(t)\}$ be a birth and death process with birth rates λ_i and death rates μ_i. Show that, when the population size is currently i, the time to the next birth is exponential with rate λ_i.

   (*Hint: First, this for a birth process. Extend to a birth and death process by superimposing the birth and death processes into one counting process and thinning back to births.*)