1. Consider a machine that is either in an operating state or a repair state. Suppose that, when it is operating, it stays that way for an exponential amount of time with rate $\lambda$ and then, when in for repair, the repair takes an exponential amount of time with rate $\mu$. Further suppose that all of these exponentials are independent.

Let

$$X(t) = \begin{cases} 0, & \text{if the machine is operating at time } t \\ 1, & \text{if the machine is in repair at time } t \end{cases}$$

Show that $\{X(t)\}$ is a birth and death process and give the birth and death rates.

2. Each individual in a biological population is assumed to give birth after an exponential amount of time with rate $\lambda$, and to die after an exponential amount of time with rate $\mu$. In addition, new individuals are immigrating in to the population according to a Poisson process with rate $\theta$, however, immigration is not allowed when the population size is $N$ or larger.

Let $X(t)$ be the number of individuals in the population at time $t$. Show that $\{X(t)\}$ is a birth and death process and give the birth and death rates.

[Hint: For any one individual, the probability it gives birth in any interval $(t, t+h]$ is $P(B \leq h)$ where $B \sim \text{exp}(\text{rate} = \lambda)$. This is due to the lack of memory property of the exponential. Show that, for small $h$, this probability is $\lambda h + o(h)$. Now this is like other problems we have done in class where the word “exponential” was not mentioned, but instead we said something like “the individual birth rate is $\lambda$”. Note that the meaning of the phrase “the individual birth rate is $\lambda$” was that the probability an individual gives birth in a small interval of length $h$ is $\lambda h + o(h)$. The same sort of thing can be said about the individual exponential death times. Okay, so now this hint is longer than the problem so that’s kind of weird.]

3. Consider a birth and death process with birth rates $\lambda_i = (i + 1)\lambda$, $i \geq 0$, and death rates $\mu_i = i\mu$, $i \geq 0$.

   (a) Determine the expected time to go from state 0 to state 4.
   (b) Determine the expected time to go from state 2 to state 5.

4. Each time a machine is repaired, it remains up and working for an exponentially distributed time with rate $\lambda$. It then fails, and its failure is either of two types. If it is a type 1 failure, then the time to repair the machine is exponentially distributed with mean $\mu_1$; if it is a type 1 failure, then the time to repair the machine is exponentially distributed with mean $\mu_2$. Each failure is, independently of the time it took the machine to fail, a type 1 failure with probability $p$ and a type 2 failure with probability $1 - p$.

   Write down the generator matrix for this birth and death process.

5. [Required for 5560 only] Let $\{X(t)\}$ be a birth and death process with birth rates $\lambda_i$ and death rates $\mu_i$. Show that, when the population size is currently $i$, the time to the next birth is exponential with rate $\lambda_i$.

   (Hint: First, this for a birth process. Extend to a birth and death process by superimposing the birth and death processes into one counting process and thinning back to births.)