APPM 4360/5360 **Homework Assignment #7 Solutions** **Spring 2019**

Problem #1 (14 points): Evaluate

$$\int_0^\infty \frac{x^3 \sin kx}{(x^2+a^2)^2}\,dx, \qquad k>0, a>0$$

Solution: Since the integrand is even function,

$$I = \int_0^\infty \frac{x^3 \sin kx}{(x^2+a^2)^2}\,dx = \frac{1}{2}\mathrm{Im} J = \frac{-i}{2}J,$$

$$J = \int_{-\infty}^\infty \frac{x^3 e^{ikx}}{(x^2+a^2)^2}\,dx.$$

For $k>0$, closing the contour in the upper half-plane and using Jordan lemma, we find

$$J = 2\pi i \sum_{n:\mathrm{Im} z_n>0} \mathrm{Res}(f(z); z_n),$$

where $f(z) = z^3 e^{ikz}/(z^2+a^2)^2$. Its only s.p. in the upper half-plane is the double pole at $z = ia$ so

$$J = 2\pi i \left.\frac{d}{dz}\frac{z^3 e^{ikz}}{(z+ia)^2}\right|_{z=ia} = 2\pi i \left.\left(\frac{z^2(3+ikz)e^{ikz}}{(z+ia)^2} - \frac{2z^3 e^{ikz}}{(z+ia)^3}\right)\right|_{z=ia} =$$

$$= \frac{2\pi i}{(2ia)^3}(ia)^2 e^{-ka}\left(2ia(3-ka) - 2ia\right) =$$

$$= \frac{\pi i}{2} e^{-ka}(2-ka).$$

Thus,

$$I = \frac{\pi}{4}(2-ka)e^{-ka}.$$

Problem #2 (14 points): Evaluate

$$\int_0^\infty \frac{\sin kx}{x(x^2+a^2)}\,dx, \qquad k>0, a>0$$

Solution: It is convenient to rewrite the integral as follows:

$$I = \int_0^\infty \frac{\sin kx}{x(x^2+a^2)}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin kx}{x(x^2+a^2)}\,dx =$$

$$= \frac{1}{2}\int_{-\infty}^\infty \frac{e^{ikx}-1}{2ix(x^2+a^2)}\,dx - \frac{1}{2}\int_{-\infty}^\infty \frac{e^{-ikx}-1}{2ix(x^2+a^2)}\,dx.$$

Changing the integration variable $x \to -x$ in the second integral on the right we find

$$\int_{-\infty}^\infty \frac{e^{-ikx}-1}{x(x^2+a^2)}\,dx = \int_{-\infty}^\infty \frac{e^{ikx}-1}{x(x^2+a^2)}\,dx,$$

so

$$I = \int_{-\infty}^\infty \frac{e^{ikx}-1}{2ix(x^2+a^2)}\,dx.$$

Consider the last integral taken over the closed contour C in complex plane, $C = [-R,R]\cup C_R$, $C_R = \{Re^{it}, 0 \le t \le \pi\}$:

$$\oint_C \frac{e^{ikz}-1}{2iz(z^2+a^2)}\,dz =$$

$$= \int_{-R}^{R} \frac{e^{ikx}-1}{2ix(x^2+a^2)}\,dx + \int_{C_R} \frac{e^{ikz}-1}{2iz(z^2+a^2)}\,dz.$$

When we take the limit $R \to \infty$, the first integral on the right-hand side becomes I while the second tends to zero since

$$\left|\int_{C_R} \frac{e^{ikz}-1}{2iz(z^2+a^2)}\,dz\right| = \left|\int_0^{\pi} \frac{(e^{ikRe^{it}}-1)Rie^{it}dt}{2iRe^{it}(R^2e^{2it}+a^2)}\right| \le$$

$$\le \int_0^{\pi} \frac{(e^{-kR\sin t}+1)dt}{2(R^2-a^2)} \le \frac{\pi}{R^2-a^2} \to_{R\to\infty} 0.$$

Thus,

$$I = \oint_C \frac{e^{ikz}-1}{2iz(z^2+a^2)}\,dz = \oint_C f(z)\,dz = 2\pi i \operatorname{Res}(f(z); ia) =$$

$$= \pi \cdot \frac{e^{-ka}-1}{ia \cdot 2ia} = \frac{\pi}{2a^2}\left(1-e^{-ka}\right).$$

(Note that $z = 0$ is a removable singular point here.)

Problem #3 (14 points): Use a rectangular contour with corners at $\pm R$ and $\pm R + i\pi/k$, with an appropriate indentation, to show that

$$\int_0^{\infty} \frac{x}{\sinh kx} dx = \frac{\pi}{4k|k|} \quad \text{for } k \ne 0, k \text{ real.}$$

Solution: Let $C_R = C_1 + C_2 + C_3 + C_4$ be the rectangular contour and C_n, $n = 1,\ldots,4$ are its sides in counterclockwise order, $C_1 = [-R, R]$. Since $\sinh kx = 0$ on the contour at $x = i\pi/k$, we must use indentation around this point. Let $k > 0$. I.e. consider instead contour $C_{R,\epsilon} = C_1 + C_2 + \tilde{C}_3 + C_4 + C_\epsilon$, where $\tilde{C}_3 = [R + i\pi/k, \epsilon + i\pi/k] \cup [-\epsilon + i\pi/k, -R + i\pi/k]$ and $C_\epsilon = \{z = i\pi/k + \epsilon e^{i\theta} | \sin\theta < 0$ (traced from right to left) and let $\epsilon \to 0$. Then, by Cauchy theorem, $\oint_{C_{R,\epsilon}} f(z)dz = 0$. Consider

$$I = \int_{-\infty}^{\infty} \frac{x}{\sinh kx} dx = \lim_{R\to\infty} \int_{C_1} f(z)dz,$$

we also have

$$\lim_{R\to\infty} \int_{\tilde{C}_3} f(z)dz =$$

$$= \lim_{R\to\infty} \left(\int_{R+i\pi/k}^{\epsilon+i\pi/k} + \int_{-\epsilon+i\pi/k}^{-R+i\pi/k}\right) \frac{z}{\sinh kz}\,dz =$$

$$= \lim_{R\to\infty} \left(\int_R^{\epsilon} + \int_{-\epsilon}^{-R}\right) \frac{x+i\pi/k}{\sinh k(x+i\pi/k)}\,dx =$$

$$= \lim_{R\to\infty} \left(\int_R^{\epsilon} + \int_{-\epsilon}^{-R}\right) \frac{x+i\pi/k}{\sinh kx \cosh i\pi}\,dx =$$

$$= \lim_{R\to\infty} \left(\int_{\epsilon}^{R} + \int_{-R}^{-\epsilon}\right) \frac{x}{\sinh kx}\,dx = I,$$

where we used that $\sinh kx$ is odd in going to the last line. Besides,

$$\lim_{R\to\infty} \int_{C_2} f(z)dz = \lim_{R\to\infty} \int_{C_4} f(z)dz = 0,$$

because there $\sinh kz = \sinh k(\pm R + iy)$, $0 \le y \le \pi/k$, so $\lim_{R\to\infty} \sinh k(\pm R + iy) = \pm\infty$. Thus, we get

$$2I + \lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)dz = 0,$$

where

$$\int_{C_\epsilon} f(z)dz = \int_0^{-\pi} \frac{i\pi/k + \epsilon e^{i\theta}}{\sinh k(i\pi/k + \epsilon e^{i\theta})} \epsilon i e^{i\theta} d\theta =$$

$$= \int_0^{-\pi} \frac{i\pi/k + \epsilon e^{i\theta}}{\cosh i\pi \sinh k\epsilon e^{i\theta}} \epsilon i e^{i\theta} d\theta \to$$

$$\to_{\epsilon\to 0} \int_0^{\pi} \frac{i\pi}{k^2 \epsilon e^{i\theta}} \epsilon i e^{i\theta} d\theta = -\frac{\pi^2}{k^2}.$$

Thus, $I = \pi^2/2k^2$, and

$$\int_0^\infty \frac{x}{\sinh kx} dx = \frac{1}{2} I = \frac{\pi^2}{4k^2}.$$

Since $\sinh(-kx) = -\sinh kx$, in general, the answer is $\frac{\pi}{4k|k|}$.

Problem #4 (14 points): Use the keyhole contour of Figure 4.3.6 to show that, on the principal branch of x^k,

$$I(a) = \int_0^\infty \frac{x^{k-1}}{(x+a)} dx = \frac{\pi}{\sin \pi k} a^{k-1}, \quad 0 < k < 1, a > 0$$

Solution: Let C be the whole closed contour, $C_\pm$ be the upper/lower sides of the keyhole. Then, in the limit as $R \to \infty$ and $\epsilon \to 0$,

$$\int_{C_+} \frac{z^{k-1}}{(z+a)} dz = I,$$

$$\int_{C_-} \frac{z^{k-1}}{(z+a)} dz = -\int_{C_+} \frac{(ze^{2\pi i})^{k-1}}{(z+a)} dz = -e^{2\pi i(k-1)} I,$$

$$|\int_{C_R} \frac{z^{k-1}}{(z+a)} dz| \le \int_0^{2\pi} \frac{R^{k-1} R d\theta}{(R-a)} \to 0,$$

and

$$\int_{C_\epsilon} \frac{z^{k-1}}{(z+a)} dz = -\int_0^{2\pi} \frac{\epsilon^{k-1} e^{i(k-1)\theta}}{(a+\epsilon e^{i\theta})} i\epsilon e^{i\theta} d\theta \to 0.$$

Thus,

$$I(1 - e^{2\pi i(k-1)}) = \oint_C \frac{z^{k-1}}{(z+a)} dz = 2\pi i \text{Res}(f(z); -a) =$$

$$= 2\pi i \; z^{k-1}\Big|_{z=-a} = 2\pi i e^{i\pi(k-1)} a^{k-1},$$

and it follows that

$$I = \frac{2\pi i e^{i\pi(k-1)}}{1 - e^{2\pi i(k-1)}} a^{k-1} = -\frac{\pi}{\sin(\pi(k-1))} a^{k-1} = \frac{\pi}{\sin \pi k} a^{k-1}.$$

Problem #5 (20 points): Verify the Argument principle in Theorem 4.4.1 for the functions:

(a) $f(z) = \frac{z^3 + a^3}{z}$, $0 < a < 1$

(b) $f(z) = \text{sech} \pi z$

where the contour is the unit circle: $|z| = 1$.

Solution:

(a) $f(z) = \frac{z^3+a^3}{z}$, $0 < a < 1$. This function has three zeros $z = ae^{i(\pi/3+2\pi n/3)}$, $n = 0,1,2$, and one pole $z = 0$ inside the contour C (the unit circle). Thus, $N - P = 3 - 1 = 2$. On the other hand,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_C \left(\frac{3z^2}{z^3+a^3} - \frac{1}{z}\right)dz =$$

$$= \frac{1}{2\pi i}\oint_C \frac{3}{z(1+a^3/z^3)} - 1 = \frac{1}{2\pi i}\oint_C \frac{3}{z}\left(1 - \frac{a^3}{z^3} + \frac{a^6}{z^6} + \ldots\right) - 1 =$$

$$= 3 - 1 = 2.$$

(b) $f(z) = \operatorname{sech}\pi z$. This function has poles where $\cosh\pi z = 0$, two of them $z = \pm i/2$ inside the contour C (the unit circle), and no zeros. Thus, $N - P = 0 - 2 = -2$. On the other hand,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_C \frac{-\pi\sinh\pi z}{\cosh\pi z}\,dz =$$

$$= -\mathrm{Res}\left(\frac{\pi\sinh\pi z}{\cosh\pi z}; i/2\right) - \mathrm{Res}\left(\frac{\pi\sinh\pi z}{\cosh\pi z}; -i/2\right) =$$

$$= -\left.\frac{\pi\sinh\pi z}{\pi\sinh\pi z}\right|_{z=i/2} - \left.\frac{\pi\sinh\pi z}{\pi\sinh\pi z}\right|_{z=-i/2} = -1 - 1 = -2.$$

Problem #6 (10 points): Use the Argument Principle to show that $f(z) = z^5 + 1$ has one zero in the first quadrant.

Solution: Since $f(z)$ is entire, by Argument Principle,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N,$$

where N is the number of zeros of $f(z)$ inside C. Let C be the contour in Fig. 4.4.3 in the first quadrant with a corner at $z = 0$. Then 1)on $[0, R]$ the argument $\arg f(z)$ does not change, 2)on the quarter-circle C,

$$\Delta_C \arg f(z) = \frac{\pi}{2}\cdot 5 = \frac{5\pi}{2},$$

and on the interval of imaginary axis $[iR, 0]$ we have $f(z) = (iy)^5 + 1 = iy^5 + 1$, so

$$\tan\arg f(z) = \frac{\mathrm{Im} f(z)}{\mathrm{Re} f(z)} = y^5 > 0,$$

it changes from $+\infty$ to 0 for large R, and therefore $\arg f(z)$ changes from $5\pi/2$ to 2π. Thus,

$$N = \frac{1}{2\pi}\Delta_{total}\arg f(z) = \frac{2\pi}{2\pi} = 1,$$

one zero indeed.

Problem #7 (20 points):

(a) Show that $e^z - (4z^2 + 1) = 0$ has exactly two roots for $|z| < 1$. Hint: in Rouché's Theorem use $f(z) = -4z^2$ and $g(z) = e^z - 1$, so that when C is the unit circle

$$|f(z)| = 4 \text{ and } |g(z)| = |e^z - 1| \le |e^z| + 1.$$

(b) Show that the improved estimate $|g(z)| \le e - 1$ can be deduced from $e^z - 1 = \int_0^z e^w\,dw$ and that this allows us to establish that $e^z - (2z + 1) = 0$, has exactly one root for $|z| < 1$.

Solution:

(a) Since on $|z|=1$, $|g(z)| = |e^z - 1| \le |e^z| + 1 \le e + 1 < 4$, we have $|f(z)| > |g(z)|$ there. So, by Rouche's theorem, $f(z)$ and $f(z)+g(z)$ have equal number of zeros inside, and this number is 2 for $f(z)$, so also for $f(z)+g(z) = e^z - (4z^2+1)$.

(b)
$$|g(z)| \le \int_0^{|z|} e^{|w|} d|w| = e^{|z|} - 1 = e - 1$$
on $|z|=1$. Since $e - 1 < 2 = |2z|$ on the unit circle, Rouche theorem yields the second claim.

Problem #8 (30 points): Find the Fourier transform of the following functions:

(a) e^{-x^2+iax}, $a>0$

(b) $\frac{x}{x^2+2ax+2a^2}$, $a>0$

Solution:

(a) e^{-x^2+iax}, $a>0$

$$\hat{F}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx =$$
$$= \int_{-\infty}^{\infty} \exp(-x^2 + iax) e^{-ikx} dx =$$
$$= e^{-((k-a)/2)^2} \int_{-\infty}^{\infty} \exp(-(x + i(k-a)/2)^2) dx =$$
$$= \sqrt{\pi} e^{-(k-a)^2/4},$$

the last step is formal but can be justified.

$$\int_{-\infty}^{\infty} \exp(-(x + i(k-a)/2)^2) dx =$$
$$= \int_{-\infty + i(k-a)/2}^{\infty + i(k-a)/2} \exp(-z^2) dz.$$

Consider a rectangular closed contour C with corners at $\pm R$ and $\pm R + i(k-a)/2$. Then $\oint_C \exp(-z^2) dz = 0$ by analyticity. Then the integrals over the upper and lower sides are equal up to the opposite sign, and the integrals over left and right sides vanish in the limit as $R \to \infty$ (there $z = \pm R + iy$, $y \in [0, (k-a)/2]$ is finite). Thus, the result.

(b) $\frac{x}{x^2+2ax+2a^2}$, $a>0$

$z^2 + 2az + 2a^2 = (z+a)^2 + a^2$, so its zeros are $z = -a \pm ia$. Then

$$\hat{F}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-\infty}^{\infty} \frac{x}{x^2 + 2ax + 2a^2} e^{-ikx} dx =$$

(for $k > 0$ we close the contour in the lower half plane and use Jordan lemma)

$$= -2\pi i \text{Res}(f(z) e^{-ikz}; z = -a - ia) = -2\pi i \frac{(-a - ia) e^{-ka+ika}}{-2ia} = -(1+i)\pi e^{-ka+ika} \qquad \text{for } k > 0;$$

and (for $k < 0$ we close the contour in the upper half plane and use Jordan lemma)

$$= 2\pi i \text{Res}(f(z) e^{-ikz}; z = -a + ia) = 2\pi i \frac{(-a + ia) e^{ka+ika}}{2ia} = (-1+i)\pi e^{ka+ika} \qquad \text{for } k < 0.$$

These results go to different (finite) limits as $k \to 0$. The discontinuity of $\hat{F}(k)$ at $k = 0$ is related to the fact that, for $k = 0$, the integral $\hat{F}(k)$ diverges.

Problem #9 (20 points): Find the Inverse Laplace transform of the following functions:

(a) $\frac{1}{s^2(s+a)}$, $a > 0$

(b) $\frac{1}{(s+b)(s^2+a^2)}$, $a > 0$, $b > 0$

Solution:

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{F}(s)e^{sx}ds,$$

we close the Bromwich contour by a large semicircle in the left half-plane for $x > 0$. (For $x < 0$, we close it by a large semicircle in the right half-plane and get $f(x) = 0$ since the integrand is analytic inside the contour.)

(a) $\hat{F}(s) = \frac{1}{s^2(s+a)}$, $a > 0$. The inverse LT is (for $x > 0$)

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s^2(s+a)}ds =$$

$$= \text{Res}\left(\frac{e^{sx}}{s^2(s+a)}; s=0\right) + \text{Res}\left(\frac{e^{sx}}{s^2(s+a)}; s=-a\right) =$$

$$= \frac{d}{ds}\frac{e^{sx}}{s+a}\bigg|_{s=0} + \frac{e^{-ax}}{(-a)^2} =$$

$$= \frac{x}{a} - \frac{1}{a^2} + \frac{e^{-ax}}{a^2}.$$

(b) $\hat{F}(s) = \frac{1}{(s+b)(s^2+a^2)}$, $a > 0$, $b > 0$. The inverse LT is (for $x > 0$)

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{(s+b)(s^2+a^2)}ds =$$

$$= \text{Res}\left(\frac{e^{sx}}{(s+b)(s^2+a^2)}; s=-b\right) + \text{Res}\left(\frac{e^{sx}}{(s+b)(s^2+a^2)}; s=ia\right) + \text{Res}\left(\frac{e^{sx}}{(s+b)(s^2+a^2)}; s=-ia\right) =$$

$$= \frac{e^{-bx}}{a^2+b^2} + \frac{e^{iax}}{2ia(ia+b)} + \frac{e^{-iax}}{-2ia(-ia+b)} = \frac{e^{-bx}}{a^2+b^2} + \frac{(b-ia)e^{iax} - (b+ia)e^{-iax}}{2ia(b^2+a^2)} =$$

$$= \frac{e^{-bx}}{a^2+b^2} + \frac{b\sin(ax)}{a(a^2+b^2)} - \frac{\cos(ax)}{(a^2+b^2)}.$$

Problem #10 (30 points): Given the differential equation for $y(t)$ and initial conditions

$$\frac{d^2y}{dt^2} + \omega^2 y = \cos t, \qquad y(0) = y'(0) = 0, \quad \omega > 0$$

(a) Take the Laplace transform of this equation and solve for the Laplace transform of y: $\hat{Y}(s)$

(b) Find the inverse Laplace transform of $\hat{Y}(s)$ when $\omega \neq 1$ thereby finding $y(t)$

(c) Find the inverse Laplace transform of $\hat{Y}(s)$ when $\omega = 1$ thereby finding $y(t)$

In this way one has solved the differential equation.

Solution:

(a) Let

$$\hat{Y}(s) = \int_0^\infty y(t) e^{-st} dt,$$

then the LT of $\frac{d^2y}{dt^2}$ is (integrating by parts twice)

$$\int_0^\infty y''(t) e^{-st} dt = -y'(0) - sy(0) + s^2 \hat{Y}(s) = s^2 \hat{Y}(s),$$

by using the initial conditions. The LT of the right-hand side of the equation is

$$\int_0^\infty \cos(t) e^{-st} dt = \frac{1}{2} \int_0^\infty (e^{(i-s)t} + e^{-(i+s)t}) dt = \frac{1}{2}\left(-\frac{1}{i-s} + \frac{1}{i+s}\right) = \frac{s}{s^2+1}.$$

Thus, the LT of the equation is

$$(s^2+\omega^2)\hat{Y}(s) = \frac{s}{s^2+1}$$

and

$$\hat{Y}(s) = \frac{s}{(s^2+\omega^2)(s^2+1)}.$$

(b) Doing inverse LT using the Bromwich contour as usual. For $t > 0$,

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{Y}(s) e^{sx} ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{se^{st}}{(s^2+1)(s^2+\omega^2)} ds =$$

$$= \mathrm{Res}\left(\frac{se^{st}}{(s^2+1)(s^2+\omega^2)}; s = i\right) + \mathrm{Res}\left(\frac{se^{st}}{(s^2+1)(s^2+\omega^2)}; s = -i\right) +$$

$$+\mathrm{Res}\left(\frac{se^{st}}{(s^2+1)(s^2+\omega^2)}; s = i\omega\right) + \mathrm{Res}\left(\frac{se^{st}}{(s^2+1)(s^2+\omega^2)}; s = -i\omega\right) =$$

$$= \frac{ie^{it}}{2i(\omega^2-1)} + \frac{-ie^{-it}}{-2i(\omega^2-1)} + \frac{i\omega e^{i\omega t}}{2i\omega(1-\omega^2)} + \frac{-i\omega e^{-i\omega t}}{-2i\omega(1-\omega^2)} =$$

$$\frac{\cos(t)}{\omega^2-1} + \frac{\cos(\omega t)}{1-\omega^2}.$$

(c) For $\omega = 1$ and again for $t > 0$,

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{Y}(s) e^{sx} ds =$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{se^{st}}{(s^2+1)^2} ds =$$

$$= \mathrm{Res}\left(\frac{se^{st}}{(s^2+1)^2}; s = i\right) + \mathrm{Res}\left(\frac{se^{st}}{(s^2+1)^2}; s = -i\right) =$$

$$= \frac{d}{ds}\frac{se^{st}}{(s+i)^2}\bigg|_{s=i} + \frac{d}{ds}\frac{se^{st}}{(s-i)^2}\bigg|_{s=-i} =$$

$$= \frac{(1+it)e^{it}}{(2i)^2} - \frac{2ie^{it}}{(2i)^3} + \frac{(1-it)e^{-it}}{(-2i)^2} - \frac{-2ie^{-it}}{(-2i)^3} =$$

$$= -\frac{\cos(t) - t\sin(t)}{2} + \frac{\cos(t)}{2} = \frac{t\sin(t)}{2}.$$

Note that this is the limit of the result in part b) as $\omega \to 1$, as it should be.

A good check of correctness of the result in part b) or c) is to verify that the equation and the initial conditions are satisfied.

Extra-Credit Problem #11 (20 points):

(a) Show that the inverse Laplace transform of $\hat{F}(s) = e^{-as^{1/2}} s$, $a > 0$, is given by

$$f(x) = 1 - \frac{1}{\pi}\int_0^\infty \frac{\sin(ar^{1/2})}{r} e^{-rx} dr$$

Note that the integral converges at $r = 0$.

(b) Use the definition of the error function integral

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-r^2} dr$$

to show that an alternative form for $f(x)$ is

$$f(x) = 1 - \text{erf}\left(\frac{a}{2\sqrt{x}}\right)$$

Solution:

(a) Here we have branch points $s = 0$ and $s = \infty$; choose the branch cut on $(-\infty; 0]$. Then choose the contour of integration for the inverse LT going around the cut, see Fig. 4.5.2 in the textbook. We use the setting and the notations in Example 4.5.3 on pp. 275–276 of the book. Again one shows that $\lim_{R\to\infty}\int_{C_{R_A}} = \lim_{R\to\infty}\int_{C_{R_B}} = 0$. However, now (on C_ϵ, $s = \epsilon e^{i\theta}$, $-\pi < \theta < \pi$)

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{C_\epsilon}\frac{e^{sx}e^{-as^{1/2}}}{s} ds = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\pi^{-\pi}\frac{e^{\epsilon x e^{i\theta}}e^{-a\epsilon^{1/2}e^{i\theta/2}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta =$$

$$= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_\pi^{-\pi} d\theta = -1.$$

Also the sum of integrals over the sides of the cut is

$$\frac{1}{2\pi i}\left(\int_{s=re^{i\pi}}\frac{e^{sx}e^{-as^{1/2}}}{s} ds + \int_{s=re^{-i\pi}}\frac{e^{sx}e^{-as^{1/2}}}{s} ds\right) = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\left(\int_\infty^\epsilon \frac{e^{-rx}e^{-ar^{1/2}i}}{r} dr + \int_\epsilon^\infty \frac{e^{-rx}e^{-ar^{1/2}\cdot(-i)}}{r} dr\right) =$$

$$= \frac{1}{2\pi i}\int_0^\infty \frac{e^{-rx}(e^{iar^{1/2}} - e^{-iar^{1/2}})}{r} dr = \frac{1}{\pi}\int_0^\infty \frac{\sin(ar^{1/2})}{r} e^{-rx} dr.$$

Now the first equation on p. 276 implies that

$$f(x) = -\frac{1}{2\pi i}\left(\lim_{\epsilon\to 0}\int_{C_\epsilon} + \int_{s=re^{i\pi}} + \int_{s=re^{-i\pi}}\right)\frac{e^{sx}e^{-as^{1/2}}}{s} ds = 1 - \frac{1}{\pi}\int_0^\infty \frac{\sin(ar^{1/2})}{r} e^{-rx} dr.$$

(b)
