APPM 5470, Methods of Applied Mathematics: Partial Differential Equations and Integral Equations, HW 7, due 12/8 *Fall 2017*

Book Problems:

Chapter 9: 3, 6, 11

Additional Problems:

A1) This problem will use methods of potential theory to find a boundary integral equation representation of the solution to the Dirichlet problem

$$\Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^3$$
$$u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega,$$

where Ω is bounded and its boundary $\partial\Omega$ is smooth. Recall the fundamental solution in $\mathbb{R}^3$: $\Phi(\mathbf{x}, \mathbf{y}) = (4\pi|\mathbf{x} - \mathbf{y}|)^{-1}$.

(a) Show that for any $\mu \in C(\partial\Omega)$, the double layer potential

$$u(\mathbf{x}) = \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y}\Phi(\mathbf{x}, \mathbf{y})\right] \mu(\mathbf{y})\, \mathrm{d}S_y \tag{1}$$

is harmonic in $\mathbb{R}^3 \setminus \partial\Omega$.

(b) Prove

$$\int_{\partial\Omega} \left[\frac{\partial}{\partial n_y}\Phi(\mathbf{x}, \mathbf{y})\right] \mathrm{d}S_y = \begin{cases} -1, & \mathbf{x} \in \Omega, \\ -\frac{1}{2}, & \mathbf{x} \in \partial\Omega, \\ 0, & \mathbf{x} \in \overline{\Omega}^c = \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases}$$

(c) For the double layer potential in Eq. (1) to solve the Dirichlet problem, we require

$$\lim_{\substack{\mathbf{z}\to\mathbf{x}\in\partial\Omega \\ \mathbf{z}\in\Omega}} u(\mathbf{z}) = f(\mathbf{x}).$$

Assuming this and using the relations you proved in (b), show that the density μ in (1) satisfies the singular boundary integral equation

$$\int_{\partial\Omega} \left[\frac{\partial}{\partial n_y}\Phi(\mathbf{x}, \mathbf{y})\right] \mu(\mathbf{y})\, \mathrm{d}S_y - \frac{1}{2}\mu(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega.$$

Integral equations like this one are at the heart of modern boundary integral numerical methods.

A2) In this problem, you will show that the Fourier transform of a constant is a delta distribution. The general strategy is to find the Fourier transform of $f_L(x) = H(L - |x|) = \begin{cases} 1, & |x| < L \\ 0, & \text{else} \end{cases}$ and then show that $\lim_{L\to\infty} \hat{f}_L(k)$ is a delta distribution. Recall the Fourier transform

$$\hat{f}(k) = \int_{\mathbb{R}} f(x) e^{ikx}\, \mathrm{d}x. \tag{2}$$

(a) Compute $\hat{f}_L(k)$.

(b) Show that $\lim_{R\to\infty} \int_{-R}^{R} \hat{f}_L(k)\, \mathrm{d}k = 2\pi$. *Hint: some basic complex variables can be handy here.*

(c) Now prove that $\hat{f}_L(k) \to 2\pi\delta(k)$ in the sense of distributions.

A3) (a) Explain mathematically the concept of finite or infinite signal propagation speed for the heat $u_t = ku_{xx}$ and wave $u_{tt} - c^2u_{xx} = 0$ equations. Assume compactly supported initial conditions on $\mathbb{R}$ and justify your answer explicitly by using the solution to each problem.

(b) Consider the "initial value problem" for Laplace's equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u(x,0) &= f(x), \quad \lim_{y\to\infty} u(x,y) = 0, \quad x \in \mathbb{R}, \end{aligned}$$

where y plays the role of "time". Assume f is smooth and compactly supported.

i. Solve this boundary value problem.

ii. Does the solution exhibit finite or infinite propagation speed? Justify your answer mathematically and explicitly as in part (a).

A4) Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x \in (0,\pi),\ y > 0\}$.

(a) Solve

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad x \in \Omega, \\ u(x,0) &= V, \quad \lim_{y\to\infty} u(x,y) = 0, \quad x \in (0,\pi), \\ u(0,y) &= u(\pi,y) = 0, \quad y > 0, \end{aligned} \tag{3}$$

using separation of variables.

(b) Why did we require $\lim_{y\to\infty} u(x,y) = 0$?

(c) Show that the solution can be written as

$$u(x,y) = \frac{4V}{\pi}\mathrm{Im}\sum_{n \text{ odd}} \frac{Z^n}{n}, \quad Z = e^{i(x+iy)}. \tag{4}$$

(d) Now sum the infinite series (the Taylor expansion $\log(1+Z) = Z - \frac{1}{2}Z^2 + \frac{1}{3}Z^3 - \frac{1}{4}Z^4 + \cdots$ may be helpful).

(e) Manipulate the summed series and recal $\mathrm{Im}\log z = \arg z$ on the principal branch of $\log z$ to show that the solution in closed form is

$$u(x,y) = \frac{2V}{\pi}\tan^{-1}\left(\frac{\sin x}{\sinh y}\right). \tag{5}$$