APPM 4360/5360 **Homework Assignment #6 Solutions** **Spring 2019**

Problem #1 (30 points): Evaluate the integral

$$I = \frac{1}{2\pi i} \oint_C f(z)\, dz,$$

where C is the unit circle centered at the origin, for the following $f(z)$:

(a) $f(z) = \dfrac{z^3}{z^4 + a^4}$, $0 < a < 1$

(b) $f(z) = \dfrac{\log(z-b)}{z^2 + a^2}$, $0 < a < 1$, $b > 1$, principal branch

(c) $\tan(2z)$

Solution:

(a) There are four singular points inside C, the roots z_j, $j = 1,2,3,4$, of $z^4 + a^4 = 0$, and no finite s.p. outside of C. So $I = \operatorname{Res}(f(z);\infty)$. Since

$$f(z) = \frac{z^3}{z^4+a^4} = \frac{1}{z(1+a^4/z^4)} = \frac{1}{z}\sum_{n=0}^{\infty}(-a^4/z^4)^n = \frac{1}{z} - \frac{a^4}{z^5} + \dots,$$

we have $I = \operatorname{Res}(f(z);\infty) = 1$. Alternatively, the residue at each of the four simple poles is $\operatorname{Res}(f(z); z_j) = z_j^3/(4z_j^3) = 1/4$, so adding them gives 1 again.

(b) This f has a branch point at $z = b$, make the cut on $[b,+\infty)$ and, for the principal branch, when $z = x + i0$, $x > b$, i.e. on the top side of the cut, we have $\log(z-b) = \log|x-b|$. Then $z = \pm ia$ are two simple poles inside C. Since we are integrating over the unit circle,

$$I = \operatorname{Res}(f;ia)+\operatorname{Res}(f;-ia) = \frac{\log(ia-b)}{2ia} + \frac{\log(-ia-b)}{-2ia} = \frac{1}{2ia}\log\frac{(-b+ia)}{(-b-ia)} = \frac{1}{2a}(\arg(-b+ia)-\arg(-b-ia)) =$$

$$= \frac{1}{2a}(\arctan(-a/b)+\pi-\arctan(a/b)-\pi) = -\frac{\arctan(a/b)}{a}.$$

(c) The singular points are those where $\cos(2z) = 0$, i.e. $z = z_k = \pi/4 + \pi k/2$, $k \in \mathbb{Z}$. Two such points, $z = \pm\pi/4$ are inside C. Using that

$$\tan(2z) = \frac{\sin(2(z_k+u))}{\cos(2(z_k+u))} = \frac{\sin(2z_k)\cos(2u)}{-\sin(2z_k)\sin(2u)} = -\frac{\cos(2u)}{\sin(2u)} =$$

$$= -\frac{1-(2u)^2/2+\dots}{2u-(2u)^3/6+\dots} = -\frac{(1-(2u)^2/2+\dots)(1+(2u)^2/6+\dots)}{2u} = -\frac{1}{2u} + \frac{2u}{3} + \dots,$$

we get

$$I = \operatorname{Res}(f;\pi/4) + \operatorname{Res}(f;-\pi/4) = -\frac{1}{2} - \frac{1}{2} = -1.$$

Or, shorter, using residue formula,

$$I = \operatorname{Res}(f;\pi/4) + \operatorname{Res}(f;-\pi/4) = \left.\frac{\sin(2z)}{2\cos'(2z)}\right|_{z=\pi/4} + \left.\frac{\sin(2z)}{2\cos'(2z)}\right|_{z=-\pi/4} =$$

$$= -\frac{1}{2} - \frac{1}{2} = -1.$$

Problem #2 (10 points): Let C be the unit circle centered at the origin. Evaluate the integral

$$I = \frac{1}{2\pi i} \oint_C f(z)\, dz,$$

for the following $f(z)$ in two ways: (i) enclosing the singular points inside C and (ii) enclosing the singular points outside C (by including the point at infinity). Show that you get the same result in both cases.

$$f(z) = \frac{z^2+1}{z^2-a^2}, \qquad a^2 < 1.$$

Solution:

$$f(z) = 1 + \frac{1+a^2}{2a(z-a)} - \frac{1+a^2}{2a(z+a)},$$

which has simple poles at $z = \pm a$.
(i) Since these poles are inside C,

$$I = \text{Res}(f;a) + \text{Res}(f;-a) = (1-1)\frac{1+a^2}{2a} = 0.$$

(ii) Since f is analytic outside C, $I = \text{Res}(f;\infty) = 0$. Both results are the same, as expected since f is rational.

Problem #3 (10 points): Let $f(z)$ be analytic outside a circle C_R enclosing the origin.

(a) Show that

$$\frac{1}{2\pi i} \oint_{C_R} f(z) dz = \frac{1}{2\pi i} \oint_{C_\rho} f(1/t) \frac{dt}{t^2},$$

where C_ρ is a circle of radius $1/R$ enclosing the origin. For $R \to \infty$ conclude that the integral can be computed to be $\text{Res}(f(1/t)/t^2;0)$.

(b) Suppose $f(z)$ has the convergent Laurent expansion

$$f(z) = \sum_{j=-\infty}^{-1} A_j z^j.$$

Show that the integral above equals A_{-1}. (See also Eq. (4.1.11).)

Solution:

(a) Let $z = 1/t$, then $dz = -dt/t^2$, and contour C_R is a *clockwise oriented* contour around $z = \infty$ or $t = 0$. If we change orientation, we get

$$\frac{1}{2\pi i} \oint_{C_R} f(z) dz = \frac{1}{2\pi i} \oint_{C_\rho} f(1/t) \frac{dt}{t^2},$$

where C_ρ is a counterclockwise oriented circle of radius $1/R$ enclosing the origin. For $R \to \infty$ we use analiticity of $f(1/t)$ inside the circle to conclude that the integral can be computed to be $\text{Res}(f(1/t)/t^2;0)$.

(b)

$$f(1/t) = \sum_{j=-\infty}^{-1} A_j (1/t)^j = \sum_{n=1}^{\infty} A_{-n} t^n,$$

so

$$\frac{1}{2\pi i} \oint_{C_\rho} f(1/t) \frac{dt}{t^2} = \sum_{n=1}^{\infty} A_{-n} \cdot \frac{1}{2\pi i} \oint_{C_\rho} t^n \frac{dt}{t^2} = A_{-1}.$$

Problem #4 (15 points): Determine the type of singular point each of the following functions have at infinity.

(a) $\frac{z^n}{z^m+a}$, $a > 0$, $n > m$ positive integers.
(b) $\log(z^2 + a^2)$, $a > 0$
(c) $\cos z$

Solution:

(a) $\frac{z^n}{z^m+a}$, $a > 0$, $n > m$ positive integers: let $z = 1/t$, then

$$\frac{z^n}{z^m + a} = \frac{t^m}{t^n(1 + at^m)},$$

which has a pole of order $n - m$ at $t = 0$ of strength 1, so pole of order $n - m$ and strength 1 at $z = \infty$.

(b) $\log(z^2 + a^2)$, $a > 0$:

$$\log(z^2 + a^2) = 2\log z + \log(1 + a^2/z^2),$$

so $z = \infty$ is a logarithmic branch point.

(c) $\cos z$: $\cos z = \sum_{n=0}^{\infty}(-1)^n z^{2n}/(2n)!$ so $z = \infty$ is essential singularity.

Problem #5 (20 points): Assume that f and g are analytic outside a circle C_R of radius R centered at the origin and

$$\lim_{|z|\to\infty} f(z) = C_1 \quad \text{and} \quad \lim_{|z|\to\infty} zg(z) = C_2,$$

where C_1 and C_2 are constants. Show that

$$\frac{1}{2\pi i}\oint_{C_R} g(z)e^{f(z)}\,dz = C_2 e^{C_1}.$$

Solution: Since f and g are analytic outside C_R,

$$\frac{1}{2\pi i}\oint_{C_R} g(z)e^{f(z)}\,dz = \text{Res}(g(z)e^{f(z)};\infty).$$

To find the residue, we deduce the following:
• Since $f(z) \to C_1$ as $z \to \infty$, $e^{f(z)} \to e^{C_1}$.
• Since $zg(z) \to C_2$ as $z \to \infty$, $g(z) \to 0$ as $z \to \infty$ and $g(z)e^{f(z)} \to 0$ as $z \to \infty$ since e^{C_1} is finite.
• If $h(z) \to 0$ as $z \to \infty$, then $\text{Res}(h(z);\infty) = \lim_{z\to\infty} zh(z)$. Thus,

$$\text{Res}(g(z)e^{f(z)};\infty) = \lim_{z\to\infty} zg(z)e^{f(z)} = C_2 e^{C_1},$$

as we wanted to show.

Problem #6 (15 points): Evaluate the following real integral:

$$\int_0^\infty \frac{x^2}{(x^2+\beta^2)^2}\,dx, \qquad \beta > 0$$

Solution: $\displaystyle\int_0^\infty \frac{x^2}{(x^2+\beta^2)^2}\,dx$, $\beta > 0$. Since integrand is an even function,

$$\int_0^\infty \frac{x^2}{(x^2+\beta^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+\beta^2)^2}\,dx.$$

Since integrand is a rational function with degree of denominator = degree of numerator +2, we close the contour by large semicircle in the upper half-plane and use residues:

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+\beta^2)^2}\,dx = \pi i \mathrm{Res}(f(z); i\beta) =$$

$$= \pi i \left.\frac{d}{dz}\frac{z^2}{(z+i\beta)^2}\right|_{z=i\beta} =$$

$$= \pi i \left.\left(\frac{2z}{(z+i\beta)^2} - \frac{2z^2}{(z+i\beta)^3}\right)\right|_{z=i\beta} =$$

$$= \pi i \left(\frac{1}{2i\beta} + \frac{-\beta^2}{4i\beta^3}\right) = \frac{\pi}{4\beta}.$$

Problem #7 (40 points): Evaluate the following real integrals:

(a) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos kx}{(x^2+a^2)(x^2+b^2)}\,dx$, $a>0$, $b>0$, $k>0$

(b) $\displaystyle\int_{0}^{\infty}\frac{\cos kx}{x^4+1}\,dx$, k real

Solution:

(a) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos kx}{(x^2+a^2)(x^2+b^2)}\,dx$, $a>0$, $b>0$, $k>0$. Consider

$$J = \oint_C f(z)dz = \oint_C \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)}\,dz,$$

$$C = [-R,R]\cup\{Re^{i\theta}, 0\le\theta\le\pi\}$$

Then, by Jordan lemma, $I = \mathrm{Re}(\lim_{R\to\infty} J)$. Since

$$J = 2\pi i(\mathrm{Res}(f(z); ia) + \mathrm{Res}(f(z); ib)) = 2\pi i\left(\frac{e^{-ka}}{2ia(b^2-a^2)} + \frac{e^{-kb}}{2ib(a^2-b^2)}\right) = \frac{\pi(be^{-ka}-ae^{-kb})}{ab(b^2-a^2)},$$

so also

$$I = I(a,b) = \frac{\pi(be^{-ka}-ae^{-kb})}{ab(b^2-a^2)}.$$

This is precisely true if $a\neq b$, but special case $a=b$ can be obtained e.g. as the limit $\lim_{b\to a}$ of the last formula i.e.

$$I(a,a) = \lim_{b\to a}\frac{\pi(be^{-ka}-ae^{-kb})}{ab(b^2-a^2)} = \pi\lim_{t\to 0}\frac{(a+t)e^{-ka}-ae^{-ka}(1-kt+\dots)}{a(a+t)t(2a+t)} = \frac{\pi(1+ka)e^{-ka}}{2a^3}.$$

(b) $\displaystyle\int_{0}^{\infty}\frac{\cos kx}{x^4+1}\,dx$, k real

$$I = \int_0^{\infty}\frac{\cos kx}{x^4+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos kx}{x^4+1}\,dx$$

Consider

$$J = \oint_C f(z)dz = \oint_C \frac{e^{ikz}dz}{z^4+1},$$

$$C = [-R,R]\cup\{Re^{i\theta}, 0\le\theta\le\pi\}$$

Then, by Jordan lemma, $I = \mathrm{Re}(\lim_{R\to\infty} J/2) = \lim_{R\to\infty} J/2$. Since

$$J = 2\pi i \left(\mathrm{Res}(f(z); e^{i\pi/4}) + \mathrm{Res}(f(z); e^{3i\pi/4})\right) =$$

$$= 2\pi i \left(\frac{e^{ik(1+i)/\sqrt{2}}}{4(e^{i\pi/4})^3} + \frac{e^{ik(-1+i)/\sqrt{2}}}{4(e^{3i\pi/4})^3}\right) =$$

$$= \frac{i\pi e^{-k/\sqrt{2}}}{2}\left(\frac{e^{ik/\sqrt{2}}}{e^{3i\pi/4}} + \frac{e^{-ik/\sqrt{2}}}{e^{i\pi/4}}\right) =$$

$$= -\frac{i\pi e^{-k/\sqrt{2}}}{2}\left(e^{i\pi/4}e^{ik/\sqrt{2}} + ie^{i\pi/4}e^{-ik/\sqrt{2}}\right) =$$

$$= \frac{(1-i)\pi e^{-k/\sqrt{2}}}{2\sqrt{2}}(1+i)(\cos(k/\sqrt{2}) + \sin(k/\sqrt{2})) =$$

$$= \frac{\pi e^{-k/\sqrt{2}}}{\sqrt{2}}(\cos(k/\sqrt{2}) + \sin(k/\sqrt{2})) =$$

$$= \pi e^{-k/\sqrt{2}} \cos\left(\frac{k}{\sqrt{2}} - \frac{\pi}{4}\right) = \pi e^{-k/\sqrt{2}} \sin\left(\frac{k}{\sqrt{2}} + \frac{\pi}{4}\right),$$

so

$$I = \frac{\pi e^{-k/\sqrt{2}} \cos(k/\sqrt{2} - \pi/4)}{2} =$$

$$= \frac{\pi e^{-k/\sqrt{2}} \sin(k/\sqrt{2} + \pi/4)}{2}.$$

Problem #8 (20 points): Show that

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x}\, dx \ = \ \frac{1}{2}\sec\left(\frac{a}{2}\right), \quad |a| < \pi.$$

Use a rectangular contour with corners at $\pm R$ and $\pm R + i$.

Solution: Let $C_R = C_1 + C_2 + C_3 + C_4$ be the rectangular contour and C_n, $n = 1, \ldots, 4$ are its sides in counterclockwise order, $C_1 = [-R, R]$. Then

$$I = \int_{-\infty}^{\infty} \frac{\cosh ax}{\cosh \pi x}\, dx =$$

$$= \lim_{R\to\infty} \int_{C_1} \frac{\cosh ax}{\cosh \pi x}\, dx = \lim_{R\to\infty} \int_{C_1} f(x)\, dx.$$

Inside C_R, the integrand $f(z)$ is analytic except points where $\cosh \pi z = 0$, i.e. $z = i/2$. Thus

$$\oint_{C_R} f(z)\, dz = 2\pi i \mathrm{Res}(f(z); i/2) =$$

$$= 2\pi i \left.\frac{\cosh az}{\pi \sinh \pi z}\right|_{z=i/2} = 2\cos(a/2).$$

On the other hand,

$$\lim_{R\to\infty} \int_{C_2} f(z)\, dz = \lim_{R\to\infty} \int_{C_4} f(z)\, dz = 0,$$

because $|a| < \pi$. Besides,

$$\lim_{R\to\infty}\int_{C_3} f(z)dz = \lim_{R\to\infty}\int_R^{-R} \frac{\cosh a(x+i)}{\cosh\pi(x+i)}\,dx =$$

$$= \lim_{R\to\infty}\int_R^{-R} \frac{\cosh ax\cosh ia + \sinh ax\sinh ia}{\cosh\pi x\cosh i\pi}\,dx =$$

$$= \cos a \lim_{R\to\infty}\int_{-R}^{R} \frac{\cosh ax}{\cosh\pi x}\,dx = I\cos a.$$

Thus,

$$\oint_{C_R} f(z)dz = 2\cos(a/2) = I + I\cos a = 2I\cos^2(a/2),$$

so $I = \sec(a/2)$ and

$$\int_0^\infty \frac{\cosh ax}{\cosh\pi x}\,dx = \frac{I}{2} = \frac{1}{2}\sec(a/2).$$

Problem #9 (20 points): Consider a rectangular contour with corners at $b \pm iR$ and $b+1 \pm iR$. Use this contour to show that

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{b-iR}^{b+iR} \frac{e^{az}}{\sin\pi z}dz = \frac{1}{\pi(1+e^{-a})},$$

where $0 < b < 1$, $|\mathrm{Im}a| < \pi$.

Solution: Let $C_R = C_1 + C_2 + C_3 + C_4$ be the clockwise rectangular contour and C_n, $n = 1,\ldots,4$ are its sides in clockwise order, $C_1 = [b-iR, b+iR]$. Then

$$I = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{b-iR}^{b+iR} \frac{e^{az}}{\sin\pi z}dz = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{C_1} f(z)dz,$$

where $f(z) = \frac{e^{az}}{\sin\pi z}$. Inside C_R, the integrand $f(z)$ is analytic except points where $\sin\pi z = 0$, i.e. only $z = 1$. Thus

$$\oint_{C_R} f(z)dz = -2\pi i\,\mathrm{Res}(f(z);1) =$$

$$= -2\pi i\left.\frac{e^{az}}{\pi\cos\pi z}\right|_{z=1} = 2ie^a.$$

On the other hand,

$$\lim_{R\to\infty}\int_{C_2} f(z)dz = \lim_{R\to\infty}\int_{C_4} f(z)dz = 0,$$

because $|\mathrm{Im}a| < \pi$. Besides,

$$\lim_{R\to\infty}\int_{C_3} f(z)dz = \lim_{R\to\infty}\int_R^{-R} \frac{e^{a(b+1+iy)}}{\sin\pi(b+1+iy)}\,i\,dy =$$

$$= -e^a\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{a(b+iy)}}{\sin\pi(b+iy)\cos\pi}\,i\,dy = e^a I.$$

Thus,

$$(1+e^a)I = \frac{2ie^a}{2\pi i} = \frac{e^a}{\pi}$$

and $I = \frac{1}{\pi(1+e^{-a})}$.

Problem #10 (20 points): Use a sector contour with radius R as in figure 4.2.6, centered at the origin with angle $0 \le \theta \le \frac{2\pi}{5}$, to find, for $a > 0$,

$$\int_0^\infty \frac{dx}{x^5 + a^5} = \frac{\pi}{5a^4\sin\frac{\pi}{5}}.$$

Solution: Contour $C = C_x + C_R + C_L$; on C_x, $z = x$, $0 \le x \le R$; on C_R, $z = Re^{i\theta}$, $0 \le \theta \le \frac{2\pi}{5}$; on C_L, $z = re^{2\pi i/5}$, $0 \le r \le R$. Then

$$I = \int_0^\infty \frac{dx}{x^5 + a^5} = \lim_{R\to\infty} \int_{C_x} f(z)dz,$$

$$\lim_{R\to\infty} \int_{C_L} f(z)dz = -\lim_{R\to\infty} \int_0^R \frac{e^{2\pi i/5}dr}{r^5 + a^5} = -e^{2\pi i/5} I,$$

$$|\int_{C_R} f(z)dz| \le \int_0^{2\pi/5} \frac{Rd\theta}{R^5 - a^5} \to_{R\to\infty} 0,$$

and, since the only s.p. inside C is $z = ae^{i\pi/5}$,

$$\oint_C f(z)dz = 2\pi i \text{Res}(f(z); ae^{i\pi/5}) = \frac{2\pi i}{5a^4 e^{4i\pi/5}}.$$

Thus,

$$I(1 - e^{2\pi i/5}) = \frac{2\pi i}{5a^4 e^{4i\pi/5}},$$

so

$$I = \frac{2\pi i}{5a^4 e^{4i\pi/5}(1 - e^{2\pi i/5})} =$$

$$= \frac{2\pi i}{5a^4(2i \sin(\pi/5))} = \frac{\pi}{5a^4 \sin(\pi/5)}.$$

Extra-Credit Problem #11 (10 points): Consider a rectangular contour C_R with corners at $(\pm R, 0)$ and $(\pm R, a)$. Show that

$$\oint_{C_R} e^{-z^2} dz = \int_{-R}^R e^{-x^2} dx - \int_{-R}^R e^{-(x+ia)^2} dx + J_R = 0,$$

where

$$J_R = \int_0^a e^{-(R+iy)^2} i dy - \int_0^a e^{-(-R+iy)^2} i dy$$

Show $\lim_{R\to\infty} J_R = 0$, whereupon we have $\int_{-\infty}^\infty e^{-(x+ia)^2} dx = \int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$, and consequently, deduce that $\int_{-\infty}^\infty e^{-x^2} \cos 2ax\, dx = \sqrt{\pi} e^{-a^2}$.

Solution: Let $C_R = C_1 + C_2 + C_3 + C_4$ be the counterclockwise rectangular contour and C_n, $n = 1, \dots, 4$ are its sides in counterclockwise order, $C_1 = [-R, +R]$. Then

$$\int_{C_1} e^{-z^2} dz = \int_{-R}^R e^{-x^2} dx,$$

$$\int_{C_2} e^{-z^2} dz = \int_0^a e^{-(R+iy)^2} i dy,$$

$$\int_{C_3} e^{-z^2} dz = \int_R^{-R} e^{-(x+ia)^2} dx = -\int_{-R}^R e^{-(x+ia)^2} dx,$$

$$\int_{C_4} e^{-z^2} dz = \int_a^0 e^{-(-R+iy)^2} i dy = -\int_0^a e^{-(-R+iy)^2} i dy,$$

and, by Cauchy theorem, $\oint_{C_R} e^{-z^2} dz = 0$, which shows the first point. We have

$$|J_R| \le |\int_0^a e^{-(R+iy)^2} i dy| + |\int_0^a e^{-(-R+iy)^2} i dy| =$$

$$= e^{-R^2} \left(\int_0^a e^{y^2} dy + \int_0^a e^{y^2} dy \right) \to_{R\to\infty} 0,$$

which proves that $\int_{-\infty}^{\infty} e^{-(x+ia)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. Finally,

$$\int_{-\infty}^{\infty} e^{-(x+ia)^2}\,dx = e^{a^2}\int_{-\infty}^{\infty} e^{-x^2} e^{-2iax}\,dx =$$

$$= e^{a^2}\int_{-\infty}^{\infty} e^{-x^2}\cos 2ax\,dx,$$

the last equality because the integral of imaginary (odd) part is zero.
