APPM 4/5560

Problem Set Six (Due Wednesday, March 6th)

1. Flip a fair coin repeatedly. Find the expected number of flips until you see the pattern HTT.

2. (a) Let $X \sim Poisson(\lambda)$. Find the moment generating function (mgf) of X.

   (b) Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$ be independent. Use moment generating functions to find the distribution of $X + Y$. (Name it!)

   (c) $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$. Let $Z = X + Y$. Find the distribution of X conditional on the event that $Z = n$. (Name it!)

3. The probability generating function (pgf) for a discrete random variable X is defined as

$$G_X(z) = \mathsf{E}[z^X] = \sum_x z^x \cdot P(X = x).$$

   Suppose that $X \sim geom_0(p)$. That is, X has the geometric distribution (the one that starts from 0) with parameter p.

   (a) Find the pgf for X. Be sure to include restrictions on z if necessary.

   (b) Let $Y \sim geom_1(p)$. That is, Y has the geometric distribution (the one that starts from 1) with parameter p.
   Use part (a) to find the pgf of Y.

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed discrete random variables with common probability generating function $G_X(z)$.

   (a) Find the probability generating function of

$$S_n := X_1 + X_2 + \cdots + X_n$$

   in terms of $G_X(z)$.

   (b) Let N be a positive integer-values random variable. Consider the sum of a random number number of the X_i:

$$S_N := X_1 + X_2 + \cdots + X_N.$$

   Find the probability generating function of S_N in terms $G_X(z)$ and the pgf of N. (Hint: Write out the expectation you want and condition on N.)

5. Suppose that $X_1, X_2, \ldots$ is an infinitely long sequence of independent Bernoulli random variables with parameter p. Let $N \sim geom_1(p)$. Use probability generating functions to find the distribution of

$$S_N := X_1 + X_2 + \cdots + X_N.$$

   (Hint: The pgf of S_N, which may not be obviously recognizable, is one you will have already encountered on this HW after a bit of manipulation!)

6. **[Required for 5560 Students Only:]** Suppose that n cards, labeled 1 through n are shuffled and then dealt out on the table in a row. We say that a "match" occurs if the card labeled i appears in the ith position.

   Let p_n be the probability that no matches occur. The goal of this problem is to find p_n.

   (a) Find a recursion that, for $n \geq 3$ relates p_n to p_{n-1} and p_{n-2}. (Hint: Condition on whether or not a match occurs in position 1.)

   (b) Consider the **generating function**

   $$G(z) = \sum_{n=1}^{\infty} z^n p_n.$$

   (Note: This looks like a probability generating function but it isn't since the p_n do not add up to 1.)

   Multiply your recursion from part (a) through by nz^{n-1} and sum over $n \geq 3$. Write the result in terms of the generating function and it's derivative. Simplify by noting that $p_0 = 0$ and $p_1 = 1/2$.

   (c) Solve your differential equation from part (b). Note that we must have $G(0) = 0$.

   (d) Expand your solution from part (c) as a power series in z. Pull off the coefficient for z^n to give a non-recursive expression for p_n.