APPM 5470, Methods of Applied Mathematics: Partial Differential Equations and Integral Equations, HW 6, due 11/17 *Fall 2017*

Book Problems:

Chapter 8: 3, 4
Chapter 9: 1, 9

Additional Problems:

A1) We say $v \in C^2(\overline{\Omega})$ is subharmonic if

$$-\Delta v(\mathbf{x}) \leq 0, \quad \mathbf{x} \in \Omega.$$

(a) Prove for subharmonic v that

$$v(\mathbf{x}) \leq ⨍_{B(\mathbf{x},r)} v(\mathbf{y})\, \mathrm{d}\mathbf{y}, \quad \text{for each ball } B(\mathbf{x},r) \subset \Omega.$$

(b) Prove that therefore $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v \equiv \phi(u)$. Prove v is subharmonic.

(d) Prove that $v \equiv |\nabla u|^2$ is subharmonic whenever u is harmonic.

A2) In this problem, you will show that the long time behavior of the solution to the initial value problem for the inhomogeneous heat equation

$$\begin{aligned} u_t(\mathbf{x},t) &= \Delta u(\mathbf{x},t) + f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \quad t > 0, \\ u(\mathbf{x},0) &= g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \end{aligned} \tag{1}$$

is a solution of Poisson's equation

$$-\Delta v(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{2}$$

where

$$\lim_{t\to\infty} u(\mathbf{x},t) = v(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{3}$$

We will assume $f \in C_c^2(\mathbb{R}^3)$, i.e., f is smooth and compactly supported.

(a) Write the solution to eq. (1) using Duhamel's principle and the three-dimensional heat kernel

$$\Phi(\mathbf{x},t) = \frac{1}{(4\pi t)^{3/2}} e^{-|\mathbf{x}|^2/(4t)},$$

and show that the contribution from the initial condition $g(\mathbf{x})$ tends to zero as $t \to \infty$.

(b) For the contribution involving $f(\mathbf{x})$ from (a), swap space and time integrals and compute the time integral exactly. This should yield a convolution of f with an appropriate kernel function.

(c) Look up the asymptotic behavior of the kernel function for large t to show that

$$u(\mathbf{x},t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, \mathrm{d}\mathbf{y} + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \quad t \to \infty, \tag{4}$$

where the notation $H(t) = \mathcal{O}(G(t))$, $t \to \infty$ means $\lim_{t\to\infty} H(t)/G(t) < \infty$. Therefore, identify $\lim_{t\to\infty} u(x,t) = v(x)$ where v solves eq. (2).