APPM 4360/5360 **Homework Assignment #5 Solutions** **Spring 2019**

Problem #1 (15 points): Consider $f_n(z)g_n(z)$ in a domain D where: $|g_n(z)| \le M$; M constant in D and $f_n(z)$ converges to zero uniformly for all $z \in D$. Prove $f_n(z)g_n(z)$ converges to zero uniformly for all $z \in D$.

Solution: By definition of uniform convergence, for any $\epsilon > 0$ there is a number $N(\epsilon)$ such that, for all $z \in D$ and all $n > N(\epsilon)$, $|f_n(z)| < \epsilon$. For a given $\epsilon > 0$, let $\epsilon_1 = \epsilon/M$. Then, for all $z \in D$ and all $n > \tilde{N}(\epsilon) = N(\epsilon_1)$, we have

$$0 \le |f_n(z)g_n(z)| = |f_n(z)||g_n(z)| \le M|f_n(z)| < M\epsilon_1 = \epsilon.$$

This proves the statement.

Problem #2 (15 points): Find the Taylor series expansion of the following functions:

(a) $z^2/(1+z^3)$, $|z| < 1$;

(b) $\cosh kz$, $k > 0$ constant;

(c) ze^{ikz^2}; $k > 0$ constant.

Solution:

(a) $z^2/(1+z^3)$, $|z| < 1$; using geometric series,

$$\frac{z^2}{1+z^3} = z^2 \sum_{n=0}^{\infty} (-z^3)^n = \sum_{n=0}^{\infty} (-1)^n z^{3n+2}.$$

(b) $\cosh kz$, $k > 0$ constant;

$$\cosh kz = \sum_{n=0}^{\infty} \frac{(kz)^{2n}}{(2n)!}.$$

(c) ze^{ikz^2}; $k > 0$ constant.

$$ze^{ikz^2} = z\sum_{n=0}^{\infty} \frac{(ikz^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n k^n z^{2n+1}}{n!},$$

where $i^n = 1$ for $n = 4m$, $i^n = i$ for $n = 4m+1$, $i^n = -1$ for $n = 4m+2$ and $i^n = -i$ for $n = 4m+3$, all $m \in \mathbb{Z}$.

Problem #3 (20 points): Given

$$F(z) = \int_{-\infty}^{\infty} f(t)e^{izt}\,dt,$$

where $f(t) = e^{\alpha_1 t}$, $t < 0$, and $f(t) = e^{-\alpha_2 t}$, $t > 0$; $\alpha_j > 0$, $j = 1,2$ constants. Find the region of the complex plane where $F(z)$ is analytic; explain. Do the same if $f(t) = te^{-\kappa t^2}$, $\kappa > 0$; explain. $F(z)$ is referred to as the Fourier transform of $f(t)$.

Solution: Let $h(z,t) = f(t)e^{izt}$. Then, for both cases mentioned,
1) $h(z,t)$ is entire function of z for all t;
2) $h(z,t)$ is continuous function of t for all z;
since the integral over t is improper, one has to verify condition 3) of the theorem about analyticity of such integrals: whether there is a function $G(t)$ such that $|h(z,t)| \le G(t)$ and $\int_{-\infty}^{\infty} G(t)\,dt < \infty$. Let $z = x + iy$. Then

$$|h(z,t)| = |f(t)||e^{izt}| = |f(t)||e^{i(x+iy)t}| = |f(t)|e^{-yt}.$$

Then, for the first given $f(t)$, one gets

$$t < 0: \quad |h(z,t)| = e^{(\alpha_1 - y)t},$$

$$t > 0: \quad |h(z,t)| = e^{-(\alpha_2+y)t},$$

so $\int_{-\infty}^{\infty} |h(z,t)|dt$ is finite if and only if $\alpha_1 - y > 0$ and $\alpha_2 + y > 0$. Thus, $F(z)$ is analytic in the horizontal strip $-\alpha_2 < \text{Im}z < \alpha_1$.

For the second given $f(t)$, $\int_{-\infty}^{\infty} |h(z,t)|dt$ is finite for all z, therefore $F(z)$ is analytic for all finite z in this case. Formally, one can consider a region $y > y_0$ in $\mathbb{C}$. Then

$$|h(z,t)| < G(t) = e^{-\kappa t^2} e^{-y_0 t},$$

and $\int_{-\infty}^{\infty} G(t)dt < \infty$, so in this region one directly applies the theorem. This is true for any finite y_0 so $F(z)$ is analytic for all finite z.

Problem #4 (20 points): Let

$$F(z) = \int_0^{\infty} f(t) e^{-zt} dt,$$

where $f(t)$ is continuous function, $|f(t)| \le Ae^{-\alpha t}$, $A > 0$, $\alpha > 0$ constants. Find the region of the complex plane where $F(z)$ is analytic; explain. $F(z)$ is referred to as the Laplace transform of $f(t)$.

Solution: Let $g(z,t) = f(t)e^{-zt}$, then $g(z,t)$ is analytic in z in $\mathbb{C}$ and continuous in t for all $t > 0$. Also we have $(z = x + iy)$

$$|g(z,t)| = |f(t)||e^{-zt}| \le Ae^{-\alpha t} e^{-xt},$$

and $\int_0^{\infty} Ae^{-(x+\alpha)t} dt$ is finite for $x > -\alpha$. Therefore $F(z)$ is analytic in the (infinite) vertical strip $\text{Re}z > -\alpha$, by the same theorem as used in the previous problem.

Problem #5 (20 points): Let $f(z) = 1/(z^2 + \alpha^2)$, $\alpha > 0$. Find the Laurent expansion in the regions

(a) $|z| > \alpha$

(b) $|z| < \alpha$

Solution:

$$f(z) = \frac{1}{z^2+\alpha^2} = \frac{1}{2i\alpha}\left(\frac{1}{z-i\alpha} - \frac{1}{z+i\alpha}\right).$$

(a) $|z| > \alpha$

$$f(z) = \frac{1}{2i\alpha z}\left(\frac{1}{1-i\alpha/z} - \frac{1}{1+i\alpha/z}\right) =$$
$$= \frac{1}{2i\alpha z}\sum_{n=0}^{\infty}\left(\frac{(i\alpha)^n}{z^n} - \frac{(-i\alpha)^n}{z^n}\right) =$$
$$= -\sum_{m=1}^{\infty} \frac{(-1)^m (2\alpha)^{2m-2}}{z^{2m}}.$$

(b) $|z| < \alpha$

$$f(z) = \frac{1}{2\alpha^2}\left(\frac{1}{1+iz/\alpha} + \frac{1}{1-iz/\alpha}\right) =$$
$$= \frac{1}{2\alpha^2}\sum_{n=0}^{\infty}\left(\frac{(-iz)^n}{\alpha^n} + \frac{(iz)^n}{\alpha^n}\right) =$$
$$= \frac{1}{\alpha^2}\sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{\alpha^{2m}}.$$

Problem #6 (15 points): Given the function

$$f(z) = \frac{2z}{(z-i)(z+2)},$$

find the Laurent series of $f(z)$ in the regions

(a) $|z| < 1$
(b) $1 < |z| < 2$
(c) $|z| > 2$

Solution: Using partial fractions, we see that

$$\frac{2z}{(z-i)(z+2)} = \frac{2/5+4i/5}{z-i} + \frac{8/5-4i/5}{z+2}.$$

(a) For $|z| < 1$,

$$\begin{aligned} f(z) &= i\frac{2/5+4i/5}{1+iz} + \frac{4/5-2i/5}{1+z/2} \\ &= \left(\frac{4-2i}{5}\right)\sum_{n=0}^{\infty}\left(-(-i)^n + \left(\frac{-1}{2}\right)^n\right) z^n. \end{aligned}$$

(b) For $1 < |z| < 2$,

$$\begin{aligned} f(z) &= \frac{2/5+4i/5}{z(1-i/z)} + \frac{4/5-2i/5}{1+z/2} \\ &= \left(\frac{2(1+2i)}{5}\right)\sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} + \left(\frac{2(2-i)}{5}\right)\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n z^n \end{aligned}$$

(c) For $|z| > 2$,

$$\begin{aligned} f(z) &= \frac{2(1/5+2i/5)}{z(1-i/z)} + \frac{2(4/5-2i/5)}{z}\left(\frac{1}{1+2/z}\right) \\ &= \left(\frac{2(1+2i)}{5}\right)\sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} + \frac{2(4/5-2i/5)}{z}\sum_{n=0}^{\infty}\frac{(-2)^n}{z^n} \\ &= \left(\frac{2(2-i)}{5}\right)\sum_{n=0}^{\infty}\left(i^{n+1} - (-2)^{n+1}\right)\frac{1}{z^{n+1}}. \end{aligned}$$

Problem #7 (45 points): Discuss all singularities of the following functions including the type of singularity: removable, pole – include order, essential, branch point, cluster, ..., that each of these functions has in the finite z-plane. For parts a,b,c,d, if the functions have a Laurent series around any of the singularities find the first two nonzero terms.

(a) $\sec z$
(b) $\frac{1}{e^z-1}$
(c) $\frac{\log z}{z(z-2)}$
(d) $\sin(1/z^2)$
(e) $\coth(1/z)$

Solution:

(a) $\sec z = 1/\cos z$. Since $\cos z$ is an entire function, the only singular points are those where $\cos(z) = 0$, i.e. $z = z_k = \pi/2 + \pi k$, $k \in \mathbb{Z}$. Around such a point, let $z = z_k + (z - z_k) = z_k + u$, then

$$\frac{1}{\cos(z)} = \frac{1}{\cos(z_k + u)} = -\frac{1}{\sin(z_k)\sin(u)} =$$

$$= \frac{(-1)^k}{\sin(u)} = \frac{(-1)^k}{u - u^3/6 + \dots} =$$

$$= \frac{(-1)^k}{u}(1 + u^2/6 + \dots) = \frac{(-1)^k}{u} + \frac{(-1)^k u}{6} + \dots,$$

where $\dots$ correspond to positive powers of $u = z - z_k$ greater than 1. I.e. $z = z_k = \pi/2 + \pi k$ is simple pole with strength $(-1)^k$.

(b) $\frac{1}{e^z-1}$. Since the denominator is an entire function, the only singular points are those where $e^z - 1 = 0$, i.e. $z = z_n = 2i\pi n$, $n \in \mathbb{Z}$. They are isolated. Around such a point, let $z = z_n + u$, then

$$\frac{1}{e^z - 1} = \frac{1}{e^{z_n}e^u - 1} = \frac{1}{e^u - 1} =$$

$$= \frac{1}{1 + u + u^2/2 + \dots - 1} = \frac{1}{u(1 + u/2 + \dots)} =$$

$$= \frac{1 - u/2 + \dots}{u} = \frac{1}{u} - \frac{1}{2} + \dots,$$

i.e. $z = z_n = 2i\pi n$ is simple pole with residue 1 (for every n).

(c) $\frac{\log z}{z(z-2)}$. Due to $\log z$, there are two branch points, $z = 0$ and $z = \infty$. A branch cut must connect them, and a branch of $\log z$ is analytic everywhere outside the cut. All points on the cut are nonisolated (jump) s.p. of the function. If the cut passes through the point $z = 2$, then $z = 2$ is a nonisolated s.p. If it does not, e.g. if the cut connects 0 and ∞ on the negative real axis, then Laurent expansion around $z = 2$ is

$$\frac{\log z}{z(z-2)} = \frac{\log 2 + \log(1 + (z-2)/2)}{(2 + (z-2))(z-2)} = \frac{(\log 2 + (z-2)/2 + \dots)(1 - (z-2)/2 + \dots)}{2(z-2)} =$$

$$= \frac{\log 2}{2(z-2)} + \frac{1 - \log 2}{4} + \dots,$$

which means that, for every branch of log, $z = 2$ is a simple pole with residue $\log 2/2$.

(d) $\sin(1/z^2)$. Since $\sin\zeta$ is an entire function of ζ, the only singular point in the finite $\mathbb{C}$ is $z = 0$, being the only one, it is isolated. For all finite $z \neq 0$, $\sin(1/z^2)$ is equal to the convergent series,

$$\sin(1/z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(z^2)^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{4n+2}(2n+1)!}.$$

This is the Laurent series around $z = 0$, which shows that $z = 0$ is essential singular point. The first two terms of the series are

$$\sin(1/z^2) = \frac{1}{z^2} - \frac{1}{6z^6} + \dots$$

(e) $\coth(1/z)$. The only s.p. are points where $\sinh(1/z) = 0$ i.e. $1/z = i\pi k$, $k \in \mathbb{Z}$, and $z = 0$. They are isolated except for $z = 0$, which is the limit of the sequence of the other points, so $z = 0$ is a cluster point. Around a point $z = z_k = -i/(\pi k)$, let $1/z = u + i\pi k$, then

$$\coth(1/z) = \frac{\cosh(i\pi k + u)}{\sinh(i\pi k + u)} = \frac{\cosh(i\pi k)\cosh u}{\cosh(i\pi k)\sinh u} =$$

$$= \frac{1 + u^2/2 + \dots}{u + u^3/6 + \dots} = \frac{1}{u} + \frac{u}{3} + \dots =$$

$$= \frac{z}{1-i\pi kz} + \frac{1-i\pi kz}{3z} + \dots = -\frac{-i/(\pi k) + (z+i/(\pi k))}{i\pi k(z+i/(\pi k))} + \frac{i\pi k(z+i/(\pi k))}{3i/(\pi k)} + \dots =$$

$$= \frac{1}{\pi^2 k^2 (z+i/(\pi k))} + \frac{i}{\pi k} + \frac{\pi^2 k^2 (z+i/(\pi k))}{3} + \dots$$

i.e. each s.p. is simple pole of strength $1/(\pi^2 k^2)$ for $k \neq 0$. ($k=0$ corresponds to $z=\infty$ which should be considered separately, then $z = 1/u$, expand around $u=0$ and get a simple pole again.)

Problem #8 (30 points): Evaluate the integral

$$I = \frac{1}{2\pi i}\oint_C f(z)\,dz,$$

where C is the unit circle centered at the origin, and $f(z)$ is given below:

(a) $f(z) = \dfrac{z^2}{z^2+a^2}$, $0 < a < 1$

(b) $f(z) = \cot(2z)$

(c) $f(z) = \dfrac{\log(z+a)}{z+1/a}$, $a > 1$, principal branch

Solution:

(a) There are singular points at $z = ia$ and $z = -ia$; both are inside the unit circle. We have

$$f(z) = \frac{z^2}{z^2+a^2} = 1 - \frac{a^2}{z^2+a^2} = 1 + \frac{ia}{2}\left(\frac{1}{z-ia} - \frac{1}{z+ia}\right),$$

therefore

$$I = \frac{1}{2\pi i}\oint_C f(z)\,dz = 0 + \frac{ia}{2}(1-1) = 0.$$

(b) The singular points are those where $\sin(2z) = 0$, i.e. $z = z_k = \pi k/2$, $k \in \mathbb{Z}$. Only one such point, $z = 0$, is inside C. Using that

$$\cot(2z) = \frac{\cos(2z)}{\sin(2z)} = \frac{1-(2z)^2/2+\dots}{2z-(2z)^3/6+\dots} = \frac{1}{2z}\left(1 - \frac{(2z)^2}{3} + \dots\right) = \frac{1}{2z} - \frac{2z}{3} + \dots,$$

where $\dots$ stand for higher powers of z. Thus, integrating powers of z, we get

$$I = \frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{1}{2z}\,dz + 0 = \frac{1}{2}.$$

(c) This f has a branch point at $z = -a$, make the branch cut on $(-\infty, -a]$ and, for the principal branch, when $z = x > a$, $\log(z-a) = \log|x-a|$. Then $z = -1/a$ is a simple pole inside C. Expanding the function in the Laurent series around $z = -1/a$, we get

$$\frac{\log(z+a)}{z+1/a} = \frac{\log(a-1/a+(z+1/a))}{z+1/a} = \frac{\log(a-1/a)+(z+1/a)/(a-1/a)+\dots}{z+1/a} = \frac{\log(a-1/a)}{z+1/a} + \frac{1}{a-1/a} + \dots,$$

where dots stand for positive powers of $z + 1/a$. Thus, deforming the contour to the small circle around $z = -1/a$, we get

$$I = \frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{\log(a-1/a)}{z+1/a}\,dz + 0 = \log(a-1/a) = \log|a-1/a|,$$

since $a - 1/a > 0 > -a$.

Problem #9 (20 points):

(a) Let $f(z) = 1 + z^2 + z^4 + \dots$, $|z| < 1$. Find a function, call it $g(z)$, that analytically continues $f(z)$ to $|z| > 1$; what can be said about $g(z)$ on $|z| = 1$ and for $|z| < 1$; explain.

(b) Consider $f(z) = \log(2(z-1))$; $z - 1 = re^{i\theta}$. Discuss/explain the analytic continuation of the function from $R_1 \to R_2 \to R_3$ where $r > 0$ and θ is in the regions: R_1: $0 \le \theta \le \pi/2$; R_2: $\pi/3 \le \theta \le 4\pi/3$; R_3: $\pi \le \theta \le 7\pi/3$.

Solution:

(a) For $|z| < 1$, $f(z)$ is a convergent geometric series and its sum is $1/(1-z^2)$. Thus, if we define $g(z) = 1/(1-z^2)$, we get $g(z) = f(z)$ for $|z| < 1$, and $g(z)$ is analytic for all $z \in \mathbb{C}$ except for $z = \pm 1$. Thus, $g(z)$ is the analytic continuation of $f(z)$ to $|z| > 1$. On $|z| = 1$, $g(z)$ is analytic (and continuous) except for $z = \pm 1$.

(b) $f(z) = \log(2(z-1)) = \log 2 + \log(z-1)$. We have to find such branches of $\log(z-1)$, $f_1(z)$ in R_1, $f_2(z)$ in R_2 and $f_3(z)$ in R_3 that an analytic function $f(z)$ can be defined in $R_1 \cup R_2 \cup R_3$ by

$$f(z) = \begin{cases} f_1(z), & z \in R_1 \\ f_2(z), & z \in R_2 \\ f_3(z), & z \in R_3 \end{cases}$$

This is possible if $f_1(z) = f_2(z)$ in $R_1 \cup R_2$ and $f_2(z) = f_3(z)$ in $R_2 \cup R_3$.
First, we have to choose $f_1(z)$ as a branch analytic in R_1. Define $f_1(z) = \log(z-1)$ for $0 < r < \infty$, $0 \le \theta \le \pi/2$, as the principal branch of $\log(z-1)$ with the branch cut on $(-\infty, 1]$. Then $f_2(z)$ can be taken e.g. as the principal branch of $\log(z-1)$ for the cut on $[1, +\infty)$, then $f(z) = f_1(z) = f_2(z)$ for $\pi/3 \le \theta \le \pi/2$. Now, to continue f_2 from R_2 to R_3, one can take $f_3(z)$ as e.g. the branch of $\log(z-1)$ for the cut outside of R_3 e.g. on the ray $[1, +\infty \cdot e^{2\pi i/3})$, such that the range of θ is $2\pi/3 \le \theta < 8\pi/3$ for this branch. Then $f(z) = f_2(z) = f_3(z)$ for $\pi \le \theta \le 4\pi/3$. The whole range of θ for $f(z)$ here exceeds 2π therefore such $f(z)$ is defined on the Riemann surface of $\log(z-1)$ but cannot be defined in $\mathbb{C}$.

Extra-Credit Problem #10 (10 points): Given the function

$$A(z) = \int_z^\infty \frac{e^{-1/t}}{t^2}\, dt,$$

Find a Laurent expansion in powers of z for $|z| > R$, $R > 0$. Why will the same procedure fail if we consider

$$E(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt$$

Solution: In fact

$$A(z) = e^{-1/t}\Big|_z^\infty = 1 - e^{-1/z},$$

therefore the Laurent expansion in powers of z for $|z| > R$, $R > 0$, is

$$A(z) = 1 - \sum_{n=0} \frac{(-1)^n}{z^n n!} = \sum_{n=1} \frac{(-1)^{n+1}}{z^n n!}.$$

Consider the ratio test for the series:

$$\frac{|a_{n+1}(z)|}{|a_n(z)|} = \frac{n!|z|^n}{(n+1)!|z|^{n+1}} = \frac{1}{(n+1)|z|} \to_{n\to\infty} 0,$$

for all z s.t. $|z| > R > 0$, so the series converges.

As for $E(z)$, iterating the identity

$$\int_z^\infty \frac{e^{-t}}{t^k}\,dt = \frac{e^{-z}}{z^k} - k\int_z^\infty \frac{e^{-t}}{t^{k+1}}\,dt,$$

starting with $k = 1$ up to $k = n$, one obtains the series expansion with the remainder term $R_n(z)$. Consider the ratio test for the series:

$$\frac{|a_{n+1}(z)|}{|a_n(z)|} = \frac{(n+1)!|z|^{n+1}}{n!|z|^n} = (n+1)z \to_{n\to\infty} \infty,$$

for all $z \neq 0$, so the series diverges.
