APPM 5720

Solutions to Problem Set Five

1. The pdf is

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)}\beta^\alpha x^{\alpha-1}e^{-\beta x}\, I_{(0,\infty)}(x).$$

The joint pdf is

$$\begin{aligned} f(\vec{x};\alpha,\beta) &= \tfrac{1}{[\Gamma(\alpha)]^n}\beta^{n\alpha}\left(\textstyle\prod_{i=1}^n x_i\right)^{\alpha-1} e^{-\beta\sum_{i=1}^n x_i}\, \textstyle\prod_{i=1}^n I_{(0,\infty)}(x_i) \\ &= \underbrace{\frac{1}{[\Gamma(\alpha)]^n}\beta^{n\alpha}}_{a(\theta)} \underbrace{\prod_{i=1}^n I_{(0,\infty)}(x_i)}_{b(\vec{x})} \exp\left[\underbrace{(\alpha-1)}_{c_1(\theta)}\underbrace{\sum_{i=1}^n \ln x_i}_{d_1(\vec{x})} \underbrace{-\beta}_{c_2(\theta)}\underbrace{\sum_{i=1}^n x_i}_{d_2(\vec{x})}\right] \end{aligned}$$

So, by "two-parameter exponential family",

$$S = (d_1(\vec{X}), d_2(\vec{X})) = (\sum_{i=1}^n \ln X_i, \sum_{i=1}^n X_i)$$

is complete and sufficient for $\theta = (\alpha,\beta)$.

(We also say that $\sum \ln X_i$ and $\sum X_i$ are "jointly" complete and sufficient for $\theta = (\alpha,\beta)$.)

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2. First, we will use the exponential family factorization to find a complete and sufficient statistic for this model.

The pdf is

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}\, I_{\{0,1,2,\ldots\}}(x).$$

The joint pdf is

$$\begin{aligned} f(\vec{x};\lambda) &= \tfrac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod (x_i!)}\, \textstyle\prod I_{\{0,1,2,\ldots\}}(x_i) \\ &= \underbrace{e^{-n\lambda}}_{a(\lambda)} \underbrace{\frac{\prod I_{\{0,1,2,\ldots\}}(x_i)}{\prod (x_i!)}}_{b(\vec{x})} \exp[\underbrace{(\ln\lambda)}_{c(\lambda)}\underbrace{(\sum x_i)}_{d(\vec{x})}] \end{aligned}$$

So, by "one-parameter exponential family", $S = d(\vec{X}) = \sum_{i=1}^n X_i$ is complete and sufficient for this model.

(a) To find the UMVUE for λ, we need to find a function of $S = \sum X_i$ that is unbiased for λ. Let's look at $\mathsf{E}[S]$.

$$\mathsf{E}[S] = \mathsf{E}[\sum X_i] = \sum \mathsf{E}[X_i] \stackrel{ident}{=} n\mathsf{E}[X_1] \stackrel{Poisson}{=} n\lambda$$

Thus, the UMVUE for λ is

$$\widehat{\lambda} = \frac{S}{n} = \frac{\sum X_i}{n} = \overline{X}.$$

(b) To find the UMVUE for $\tau(\lambda) = e^{-\lambda}$, we need to find a function of $S = \sum X_i$ that is unbiased for λ If you can guess one and verify that it is unbiased for $e^{-\lambda}$, great. If not, try the Rao-Blackwell Theorem.

The Rao-Blackwell Theorem says (among other things) that if T is unbiased for $\tau(\lambda)$ and S is sufficient, then $T^* := \mathsf{E}[T|S]$ is still unbiased for $\tau(\lambda)$. Furthermore, T is a function of S. While we didn't have completeness back then and weren't talkng about UMVUEs, if we also have that S is complete then we have the UMVUE snce we have an unbiased estimator of $\tau(\lambda)$ that is a function of the complete and sufficient statistic S!

$e^{-\lambda}$, hmmm.... where have we seen this in relaton to the Poisson distribution... Isn't it part of the pdf?

$$P(X_1 = x) = \frac{e^{-\lambda}\lambda^x}{x!} I_{\{0,1,2,\ldots\}}(x)$$

We can see that $e^{-\lambda} = P(X_1 = 0)$. Also, we can always get an unbiased estimator for a probability out of an indicator:

$$\mathsf{E}[I_{X_1=0}] = P(X_1 = 0)$$

so we will take $I = I_{X_1=0}$.

Our UMVUE will be $\mathsf{E}[I_{X_1=0}|S]$. For "concreteness" we will compute this by first fixing $S = s$.

$$\begin{aligned}
\mathsf{E}[\widehat{\tau_1(\lambda)}|S=s] &= \mathsf{E}[I_{\{X_1=0\}}|S=s] \\
&= P(X_1 = 0|S = s) \\
&= \tfrac{P(X_1=0,S=s)}{S=s} \\
&= \tfrac{P(X_1=0,\sum_{i=1}^n X_i=s)}{P(\sum_{i=1}^n X_i=s)}
\end{aligned}$$

We would like to be able to break apart the probability in the numerator but, currently, there is an X_1 in both terms so they are not independent. Since we already known that $X_1 = 0$, we can write the other event, which says that the sum of all of the X_i is s as just that the sum of the remaining $X_2, X_3, \ldots, X_n$ is $s - 0 = s$. Thus, we have

$$\begin{aligned}
\mathsf{E}[\widehat{\tau_1(\lambda)}|S=s] &= \tfrac{P(X_1=0,\sum_{i=2}^n X_i=s)}{P(\sum_{i=1}^n X_i=s)} \\
&\overset{indep}{=} \tfrac{P(X_1=0)\,P(\sum_{i=2}^n X_i=s)}{P(\sum_{i=1}^n X_i=s)}
\end{aligned}$$

The sum of the Poissons in the denominator has again the Poisson distribution with rate $n\lambda$. The sum of the Poissons in the numerator has the Poisson distribution with rate

$(n-1)\lambda$. Thus,

$$\begin{aligned}
\mathsf{E}[\widehat{\tau_1(\lambda)}|S=s] &= \frac{P(X_1=0)\,P(\sum_{i=2}^{n} X_i=s)}{P(\sum_{i=1}^{n} X_i=s)} \\
&= \frac{\frac{e^{-\lambda}\lambda^0}{0!}\cdot\frac{e^{-(n-1)\lambda}[(n-1)\lambda]^s}{s!}}{\frac{e^{-n\lambda}[n\lambda]^s}{s!}} \\
&= \left(\frac{n-1}{n}\right)^s.
\end{aligned}$$

Removing the "concrete s" we have that the UMVUE for $\tau(\lambda)=e^{-\lambda}$ is

$$\widehat{\tau(\lambda)} = \mathsf{E}[\widehat{\tau_1(\lambda)}|S] = \left(\frac{n-1}{n}\right)^S = \left(\frac{n-1}{n}\right)^{\sum X_i}.$$

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3. The pdf is

$$f(x;\theta)=\theta x^{\theta-1}\,I_{(0,1)}(x).$$

The joint pdf is

$$\begin{aligned}
f(\vec{x};\theta) &= \theta^n \prod_{i=1}^n \left[x_i^{\theta-1}\right] \prod_{i=1}^n I_{(0,1)}(x_i) \\
&= \theta^n \left[\prod_{i=1}^n x_i\right]^{\theta-1} \prod_{i=1}^n I_{(0,1)}(x_i) \\
&= \underbrace{\theta^n}_{a(\theta)} \underbrace{\prod_{i=1}^n I_{(0,1)}(x_i)}_{b(\vec{x})} \exp[\underbrace{(\theta-1)}_{c(\theta)} \underbrace{\sum_{i=1}^n \ln x_i}_{d(\vec{x})}]
\end{aligned}$$

So, by "one-parameter exponential family",

$$S=\sum_{i=1}^n \ln X_i$$

is complete and sufficient for θ.

(a) We need to find a function of S that is unbiased for $1/\theta$. Letting $y=\ln x=g(x)$, a simple "g-inverse" transformation shows us that

$$f_Y(y)=\theta e^{\theta y}\,I_{(-\infty,0)}(x).$$

This is similar to an exponential distribution. To make it an actual exponential distribution (easier to work with), we will instead take $Y=-\ln X$. Now $Y\sim exp(rate=\theta)$. Thus,

$$S=-\sum_{i=1}^n lnX_i=-W$$

where $W\sim\Gamma(n,\theta)$.
So,

$$\mathsf{E}[S]=-\mathsf{E}[W]=-\frac{n}{\theta}$$

which implies that

$$\widehat{\theta} = -\frac{1}{n}S = -\frac{1}{n}\ln X_i$$

is the UMVUE for θ.

(b) We now need to find a function of S whose expected value is $\tau(\theta) = (\theta/(\theta+1))^n$. Note that this looks like the moment generating function of a $\Gamma(n,\theta)$, evaluated at $t=-1$. Indeed,

$$\mathsf{E}[e^S] = \mathsf{E}[e^{-W}] = M_W(-1) = \left(\frac{\theta}{\theta+1}\right)^n.$$

Thus,

$$\widehat{\tau(\theta)} = e^S = e^{\sum \ln X_i} = \prod X_i$$

is the UMVUE for $\tau(\theta)$.

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4. First, we will show that T is sufficient.

Fix any $\vec{x}$. Let $t = t(\vec{x})$.

Let $\vec{y_t}$ be any vector that maps to t, under $t(\cdot)$. Note then that $t(\vec{y_t}) = t(\vec{x})$. Thus, by the given property, we must have that

$$\frac{f(\vec{x};\theta)}{f(\vec{y_t};\theta)} = \frac{f(\vec{x};\theta)}{f(\vec{y_{t(\vec{x})}};\theta)}$$

is "θ-free". We will write

$$\frac{f(\vec{x};\theta)}{f(\vec{y_{t(\vec{x})}};\theta)} = \tilde{h}(\vec{x},\vec{y_t})$$

for some function $\tilde{h}(\cdot,\cdot)$.

Note that $\vec{y}_{t(\vec{x})}$ is a function of $t(\vec{x})$, which is a function of $\vec{x}$, so we can actually say that

$$\frac{f(\vec{x};\theta)}{f(\vec{y_{t(\vec{x})}};\theta)} = h(\vec{x})$$

for some function $h(\cdot)$.

So, we have

$$f(\vec{x};\theta) = h(\vec{x})\underbrace{f(\vec{y_{t(\vec{x})}};\theta)}_{g(t(\vec{x});\theta)}$$

and, by the Factorization Criterion for sufficiency, we have that $T = t(\vec{X})$ is sufficient for the model.

Now let's show that $T = t(\vec{X})$ is minimal sufficient.

Suppose that $S = s(\vec{X}$ is sufficient for the model. Then we can write

$$f(\vec{x};\theta) = h(\vec{x})g(s(\vec{x});\theta)$$

for some functions h and g.

So, we have, for any $\vec{x}$ and $\vec{y}$, that

$$\frac{f(\vec{x};\theta)}{f(\vec{y};\theta)} = \frac{h(\vec{x})g(s(\vec{x});\theta)}{h(\vec{y})g(s(\vec{y});\theta)}. \tag{1}$$

We want to show that T is a function of S. Suppose that $\vec{x}$ and $\vec{y}$ are such that $s(\vec{x}) = s(\vec{y})$. By (1), we then have

$$\frac{f(\vec{x};\theta)}{f(\vec{y};\theta)} = \frac{h(\vec{x})}{h(\vec{y})},$$

which is "θ-free". Thus, by the assumption of the problem, we have that $t(\vec{x}) = t(\vec{y})$.

Now

$$s(\vec{x}) = s(\vec{y}) \qquad \Rightarrow \qquad t(\vec{x}) = t(\vec{y})$$

$$\Downarrow$$

$$t(\cdot) \text{ is a one-to-one function of } s(\cdot).$$

Since $S = s(\vec{X})$ was an arbitrary sufficient statistic and T is a function of S, we have that T is minimal sufficient!

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5. Note that this distribution can't be a two-parameter exponential family because it does not have two parameters!

   (a) Use the Factorization Criterion for sufficiency.

   (b) Show that $\mathsf{E}[2(\sum X_i)^2 - (n+1)\sum X_i^2] = 0$.
   Since $2(\sum X_i)^2 - (n+1)\sum X_i^2 \neq 0$, we have exhibited a g for which $\mathsf{E}[g(S)] = 0 \not\Rightarrow g(S) = 0$. Thus, S is not complete.

   (c) This is an easy application of Problem 4.