APPM 4/5560

Problem Set Five (Due Wednesday, February 20th)

1. Consider the Markov chain on $S = \{0, 1, 2\}$ running according to the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array} \right\| \end{array}$$

   (a) Starting in state 0, what is the mean time that the process spends in state 1 prior to first hitting state 2?

   (b) Starting in state 0, what is the mean time that the process spends in state 1 prior to returning to state 0?

2. The condition of a stretch of road is classified as either being dry (D) or wet (W). For simplicity, assume that the condition of the road stays the same throughout a day and from day to day forms a 2 state Markov chain with the following transition matrix.

$$\mathbf{P} = \begin{array}{c} \\ W \\ D \end{array} \begin{array}{c} \begin{array}{cc} W & D \end{array} \\ \left\| \begin{array}{cc} 0.4 & 0.6 \\ 0.2 & 0.8 \end{array} \right\| \end{array}$$

   Accidents sometimes occur along the road. Suppose that, if the road is dry, the probability of at least one accident during the day is 0.1. If the road is wet, the probability of at least one accident during the day is 0.2.

   (a) Assuming that initially the road condition is dry and no accident occurred that day, find the expected number of days until an accident occurs.

   (b) Assuming that initially the road condition is dry and no accident occurred that day,find the probability that the condition of the road on the first day an accident occurs is also dry.

3. At each time point $n = 0, 1, 2, \ldots$, a taxi may arrive at a taxi stand at a certain hotel with probability 1/2. The taxi stand can only hold 2 taxis. If there is space for the arriving taxi, it enters. Otherwise, it leaves.

   At each of these discrete time points $n = 0, 1, 2, \ldots$, a single customer arrives with probability 1/3 looking for a taxi. If one is present (or arriving at the same time), the taxi and customer depart. If no taxis are there, the customer departs on foot.

   For $n = 0, 1, 2, \ldots$, let X_n denote the number of taxis at time $n + 0.5$.

   (a) The process is Markov. Find the long run proportion of time that no taxi is at the stand.

   (b) Suppose that, at time 0, there is 1 taxi at the stand. Find the expected number of time steps until a person arrives to find no taxi. That is, define

   $$T = \min\{n \geq 1 : \text{person arrives at time } n \text{ and finds no taxi}\}.$$

   Find $\mathsf{E}[T|X_0 = 1]$.

4. **Simulation** For the following problems, hand in all code.

   Consider a Markov chain on $\{0, 1\}$ with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 3/5 & 2/5 \\ 3/4 & 1/4 \end{array} \right\| \end{array}$$

   (a) Find the stationary distribution for this Markov chain analytically. (Not simulation.)

   (b) **Limiting Distribution:** Simulate MANY sample paths of this chain, starting from state 0 forward for "a long time". Report the proportion of time your simulation ending state 0 and ends in state 1. Do these proportions match your distribution from part (a)?

   (c) **Stationary Distribution:** Draw (simulate) a starting value from the distribution you found in part (a). Simulate forward one time step. Repeat this many times. Report the proportion of time your simulation ends in state 0 and ends in state 1. Do these proportions match your distribution from part (a)?