APPM 5470, Methods of Applied Mathematics: Partial Differential Equations and Integral Equations, HW 5, due 11/3 *Fall 2017*

Book Problems:

Chapter 6: 4, 8
Chapter 7: 1, 2

Additional Problems:

A1 (Spring 2012 prelim) Consider the forced wave equation

$$\begin{aligned} &u_{tt} - u_{xx} = e^{-x}, \quad x \in (0, L), \quad t > 0, \\ &u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad x \in (0, L), \\ &u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0. \end{aligned}$$

(a) Find a formal solution in terms of the function $f(x)$.

(b) Find conditions on f that guarantee that the expression you found in (a) is a classical solution of the system, i.e., guaranteeing that $u \in C^2([0, L], [0, \infty))$.

A2 Consider the heat equation modeling a rod with zero heat flux at its ends (Neumann boundary conditions)

$$\begin{aligned} &u_t = k u_{xx}, \quad x \in (0, L), \quad t > 0 \\ &u(x, 0) = f(x), \quad x \in (0, L), \\ &u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad t > 0. \end{aligned}$$

(a) Solve this initial/boundary value problem using separation of variables.

(b) What are the minimal conditions on f to guarantee a classical solution?

(c) What is $\lim_{t\to\infty} u(x, t)$?

(d) Find the solution when $f(x) = x^2(1 - x^2)$.

A3 Consider the Sturm-Liouville operator $\mathcal{L}$ acting on $C^2[a, b]$ functions defined according to

$$\mathcal{L}v = -(p(x)v')' + q(x)v, \quad x \in (a, b),$$

where $p \in C^1(a, b)$, $q \in C(a, b)$ and $p(x) > 0$, $q(x) \geq 0$. Assuming the standard $L^2(a, b)$ inner product $\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$ (for real-valued f and g), show that $\mathcal{L}$ is a symmetric Sturm-Liouville operator for the boundary conditions:

(a) Dirichlet: $v(a) = 0 = v(b)$.

(b) Neumann: $v'(a) = 0 = v'(b)$.

(c) Robin: $v'(a) - \alpha_a v(a) = 0 = v'(b) + \alpha_b v(b)$.

(d) What are the conditions on the constants α, β, γ, and δ for the boundary conditions

$$\alpha v(a) + \beta v'(a) - v(b) = 0, \quad \gamma v(a) + \delta v'(a) - v'(b) = 0,$$

so that the operator $\mathcal{L}$ is symmetric?

A4 Prove the comparison principle for the heat equation: If u and v are two smooth solutions of the heat equation, and if $u \leq v$ for $t = 0$, $x = 0$, and $x = L$, then $u \leq v$ for $t \in [0, \infty)$, $x \in [0, L]$.