Problem #1 (20 points): Evaluate the integral $\oint_C f(z)dz$ where C is the unit circle enclosing the origin and $f(z)$ is given by

(a) $\log(z+2)$

(b) $1/(z^2+1/4)$

Solution:

(a) $\log(z+2)$.
Consider an analytic branch of $\log(z+2)$ such that branch cut joining $z=-2$ and ∞ does not cross the unit circle centered at $z=0$. Then $\log(z+2)$ is analytic inside C and, by Cauchy theorem, $\oint_C \log(z+2)dz = 0$.

(b) $1/(z^2+1/4)$.

$$\frac{1}{z^2+1/4} = \frac{1}{i(z-i/2)} - \frac{1}{i(z+i/2)},$$

$z=\pm i/2$ are the singularities of $f(z)$ inside the contour. For each summand, we find

$$\oint_C \frac{1}{i(z-i/2)}dz = 2\pi i/i = 2\pi, \qquad \oint_C \frac{1}{i(z+i/2)}dz = 2\pi i/i = 2\pi,$$

so $\oint_C f(z)dz = 2\pi - 2\pi = 0$.

Problem #2 (20 points): Evaluate the integral $\oint_C f(z)\,dz$ where C is the unit circle centered at the origin for the following $f(z)$:

(a) $\dfrac{e^{iz}}{z}$

(b) $\dfrac{\cos z - 1}{z^3}$

Solution: Here the only singular point inside the unit circle is $z=0$. We expand the numerators in Taylor series around zero and use the integration of powers formula:

(a)

$$\frac{e^{iz}}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} = i\sum_{k=-1}^{\infty}\frac{(iz)^k}{(k+1)!},$$

power z^{-1} corresponds to $k=-1$, thus $\oint_C f(z)\,dz = 2\pi i$.

(b)

$$\frac{\cos z - 1}{z^3} = \frac{1}{z^3} - \frac{1}{2z} + \frac{z}{24} + \cdots,$$

so $\oint_C f(z)\,dz = -2\pi i/2 = -i\pi$.

Problem #3 (20 points): Evaluate the integrals $\oint_C f(z)dz$ over a contour C, where C is the boundary of a square with diagonal opposite corners at $z=-(1+i)R$ and $z=(1+i)R$, where $R>a>0$, and where $f(z)$ is given by the following (use Eq. (1.2.19) as necessary):

(a) $\dfrac{e^z}{(z-\frac{\pi i}{4}a)^2}$

(b) $\dfrac{z^2}{2z+a}$

Solution:

(a) $\dfrac{e^z}{(z-\frac{\pi i}{4}a)^2}$.

Let $z_0 = \frac{\pi i}{4}a$, it is inside the square; we expand e^z in Taylor series around $z = z_0$ here,

$$\frac{e^z}{(z-\frac{\pi i}{4}a)^2} = \frac{e^{z_0}e^{z-z_0}}{(z-z_0)^2} = \frac{e^{z_0}}{(z-z_0)^2}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!} =$$

$$= \frac{e^{z_0}}{(z-z_0)^2}\left(1+(z-z_0)+\frac{(z-z_0)^2}{2}+\dots\right) = e^{z_0}\left(\frac{1}{(z-z_0)^2}+\frac{1}{z-z_0}+\frac{1}{2}+\dots\right),$$

and the only singular point $z = z_0$ is inside the contour. Deforming the contour to a circle around $z = z_0$ and using Cauchy theorem, we find

$$\oint_C \frac{e^z}{(z-\frac{\pi i}{4}a)^2} = e^{z_0}\oint_C \frac{1}{z-z_0}\,dz = 2\pi i e^{z_0} = 2\pi i e^{\frac{\pi i}{4}a}.$$

(b) $\dfrac{z^2}{2z+a}$.

$$\frac{z^2}{2z+a} = \frac{(-a/2+(z+a/2))^2}{2(z+a/2)} =$$

$$= \frac{a^2}{8(z+a/2)} - \frac{a}{2} + \frac{z+a/2}{2},$$

and the only singular point $z = -a/2$ is inside the contour. Deforming the contour to a circle around $z = -a/2$ and using Cauchy theorem, we find

$$\oint_C \frac{z^2}{2z+a}\,dz = \oint_C \frac{a^2}{8(z+a/2)}\,dz = \frac{\pi i a^2}{4}.$$

Problem #4 (25 points): Let $f(z)$ be an entire function, with $|f(z)| \le C|z|$ for all z, where C is a constant. Show that $f(z) = Az$, where A is a constant.

Solution: Ussing the (generalized) Cauchy formula,

$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta,$$

where $C = \{|\zeta - z| = R\}$ is the circle of radius R around z in ζ-plane. Then

$$|f'(z)| \le \frac{1}{2\pi}\oint_C \frac{|f(\zeta)|}{|\zeta-z|^2}\,|d\zeta| \le$$

$$\le \frac{1}{2\pi}\int_0^{2\pi}\frac{C(|z|+R)}{R^2}\,R d\theta = C(1+|z|/R) \to_{R\to\infty} C,$$

so $f'(z)$ is entire and bounded, so it is constant by Liouville theorem. Let $f'(z) = A$, then $f(z) = Az + B$, where A, B are constants. But, since $|f(z)| \le C|z|$ for all z, taking $|z| \to 0$, we get $B = 0$. Thus, $f(z) = Az$ as claimed.

Problem #5 (20 points): Discuss whether the sequence $\{1/(nz)^2\}_1^\infty$ converges and whether the convergence is uniform for: $0 < \alpha < |z| < 1$. Discuss whether the convergence is uniform if $\alpha = 0$.

Solution:

$$\lim_{n\to\infty} \frac{1}{(nz)^2} = \frac{1}{z^2} \lim_{n\to\infty} \frac{1}{n^2} = 0,$$

so the sequence converges pointwise for every z. If $|z| > \alpha > 0$, then

$$\left| \frac{1}{(nz)^2} \right| = \frac{1}{n^2|z|^2} \le \frac{1}{\alpha^2 n^2},$$

which is a convergent numerical sequence. Thus, the convergence is uniform for $0 < \alpha < |z|$. However, for $\alpha = 0$ convergence is not uniform since $1/|z|^2$ is unbounded above in this case.

Problem #6 (20 points): Show that the following series converge uniformly in the given region:

(a) $\sum_{n=1}^{\infty} z^{2n}$, $0 \le |z| < R < 1$

(b) $\sum_{n=1}^{\infty} e^{-2nz}$, $R < \mathrm{Re} z < 1$

Solution:

(a)

$$\left| \sum_{n=1}^{\infty} z^{2n} \right| \le \sum_{n=1}^{\infty} |z|^{2n} \le \sum_{n=1}^{\infty} R^{2n} = \frac{R^2}{1-R^2},$$

i.e. the series is bounded above by a convergent numerical series which means uniform convergence by Weierstrass M-test.

(b)

$$\left| \sum_{n=1}^{\infty} e^{-2nz} \right| \le \sum_{n=1}^{\infty} |e^{-2nz}| = \sum_{n=1}^{\infty} e^{-2n\mathrm{Re} z} < \sum_{n=1}^{\infty} e^{-2nR} = \frac{e^{-2R}}{1-e^{-2R}}$$

for $R > 0$, i.e. the series is bounded above by a convergent numerical series for $R > 0$ which means uniform convergence by Weierstrass M-test for $R > 0$ (but not for $R \le 0$).

Problem #7 (20 points): Find the radius of convergence of the series $\sum_0^{\infty} a_n(z)$ where $a_n(z)$ is given by:

(a) $(-z^2)^n$

(b) $n^{2n} z^{4n}$

Solution: We apply the ratio test.

(a) $(-z^2)^n$,

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{(-z^2)^n}{(-z^2)^{n+1}} \right| = \frac{1}{|z|^2},$$

therefore the series converges for $|z| < 1$ and radius of convergence $R = 1$.

(b) $n^{2n} z^{4n}$

$$\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{n^{2n} z^{4n}}{(n+1)^{2(n+1)} z^{4(n+1)}} \right| =$$

$$= \frac{1}{(n+1)^2 (1+1/n)^{2n} |z|^4} \to_{n\to\infty} 0,$$

which shows that $R = 0$ (series converges only for $z = 0$).

Problem #8 (15 points): Find Taylor series expansions around $z = 0$ of the following functions in the given regions:

(a) $\frac{z}{1+z^2}$, $|z|<1$

(b) $\frac{\sin z}{z}$, $0<|z|<\infty$

(c) $\frac{e^{z^2}-1-z^2}{z^3}$, $0<|z|<\infty$

Solution:

(a) $\frac{z}{1+z^2}$, $|z|<1$

$$\frac{z}{1+z^2} = z\sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n+1}.$$

(b) $\frac{\sin z}{z}$, $0<|z|<\infty$

$$\frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n+1)!}.$$

(c) $\frac{e^{z^2}-1-z^2}{z^3}$, $0<|z|<\infty$

$$\frac{e^{z^2}-1-z^2}{z^3} = \frac{1}{z^3}\left(\sum_{n=0}^{\infty}\frac{z^{2n}}{n!}-1-z^2\right) =$$

$$= \sum_{n=2}^{\infty}\frac{z^{2n-3}}{n!} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(n+2)!}.$$

Problem #9 (20 points): Use the Taylor series for $(1+z)^{-1}$ about $z=0$ to find the Taylor series of $\log(1+z)$ about $z=0$ for $|z|<1$.

Solution: The Taylor series for $(1+z)^{-1}$ is just the geometric series

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n,$$

and we know that it converges uniformly in $|z|<1$. Since $\int(1+z)^{-1}\,dz = \log(1+z)+c$ and since the above series converges uniformly, we can integrate it termwise. Taking the (principal) branch of log such that $\log 1 = 0$, we find

$$\log(1+z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}.$$

Problem #10 (20 points): Find a series representation for $1/(1+z^2)$ for $|z|>1$. (Hint: see the discussion and hint of problem 3.2.8)

Solution: The Taylor series for $1/(1+z^2)$ is just the geometric series

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{2n},$$

and we know that it converges in $|z|<1$. For $|z|>1$, $1/|z|<1$, so we have

$$\frac{1}{1+z^2} = \frac{1}{z^2(1+1/z^2)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2(n+1)}}.$$

Extra-Credit Problem #11 (10 points): In Cauchy's Integral Formula (Eq. (2.6.1)), take the contour to be a circle of unit radius centered at the origin. Let $\zeta = e^{i\theta}$ to deduce

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z} d\theta$$

where z lies inside the circle. Explain why we have

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/\bar{z}} d\theta$$

and use $\zeta = 1/\bar{\zeta}$ to show

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(\frac{\zeta}{\zeta - z} \pm \frac{\bar{z}}{\bar{\zeta} - \bar{z}}\right) d\theta$$

whereupon, using the plus sign

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1-|z|^2}{|\zeta - z|^2} d\theta$$

(a) Deduce the "Poisson formula" for the real part of $f(z)$: $u(r,\phi) = \mathrm{Re} f(z)$, $z = re^{i\phi}$

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\frac{1-r^2}{(1-2r\cos(\phi-\theta)+r^2)} d\theta$$

where $u(\theta) = u(1,\theta)$.

(b) If we use the minus sign in the formula for $f(z)$ above, show that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left[\frac{1+r^2-2re^{i(\theta-\phi)}}{(1-2r\cos(\phi-\theta)+r^2)}\right] d\theta$$

and by taking the imaginary part

$$v(r,\phi) = C + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\frac{r\sin(\phi-\theta)}{(1-2r\cos(\phi-\theta)+r^2)} d\theta$$

where $C = \frac{1}{2\pi}\int_0^{2\pi} v(1,\theta)d\theta = v(r=0)$. (This last relationship follows from the Cauchy Integral formula at $z=0$ – see the first equation in this exercise.)

Solution: First formula is due to $d\zeta = ie^{i\theta}d\theta = i\zeta d\theta$. Since z is inside the unit circle, $\bar{z}$ is also and then $1/\bar{z}$ is ouside which yields the second formula by Cauchy theorem. Using that $\zeta\bar{\zeta} = 1$ on the unit circle, the third formulas are obtained by adding/subtracting the first two formulas, respectively. The fourth formula is straightforward (again use $\zeta\bar{\zeta} = 1$). Let $f(z) = u + iv$, u and v real.

(a) Take the real part of the fourth formula: $\mathrm{Re} f(z) = u(r,\phi)$, $\mathrm{Re} f(\zeta) = \mathrm{Re} f(e^{i\theta}) = u(1,\theta) = u(\theta)$, $|z| = r$ and

$$|\zeta - z|^2 = (\zeta - z)(\bar{\zeta} - \bar{z}) = 1 - (\zeta\bar{z} + z\bar{\zeta}) + r^2 = 1 - r(e^{i\theta}e^{-i\phi} + e^{i\phi}e^{-i\theta}) + r^2 = 1 - 2r\cos(\theta-\phi) + r^2.$$

Thus, Poisson formula is obtained.

(b) We have

$$\frac{\zeta}{\zeta - z} - \frac{\bar{z}}{\bar{\zeta} - \bar{z}} = \frac{1+r^2-2e^{i\theta}re^{-i\phi}}{|\zeta - z|^2} = \frac{1+r^2-2re^{i(\theta-\phi)}}{1-2r\cos(\theta-\phi)+r^2},$$

thus,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left[\frac{1+r^2-2re^{i(\theta-\phi)}}{1-2r\cos(\phi-\theta)+r^2}\right] d\theta.$$

Taking the imaginary part of the last formula, we get

$$v(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} v(1,\theta)\left[\frac{1+r^2-2r\cos(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\right]d\theta + \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left[\frac{2r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\right]d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} v(1,\theta)d\theta + \frac{1}{\pi}\int_0^{2\pi} u(\theta)\left[\frac{r\sin(\phi-\theta)}{1-2r\cos(\phi-\theta)+r^2}\right]d\theta,$$

which is the last claimed formula.
