Problem #1 (20 points): Evaluate the integral \( \oint_C f(z) \, dz \) where \( C \) is the unit circle enclosing the origin and \( f(z) \) is given by

(a) \( \log(z + 2) \)

(b) \( 1/(z^2 + 1/4) \)

Solution:

(a) \( \log(z + 2) \).

Consider an analytic branch of \( \log(z + 2) \) such that branch cut joining \( z = -2 \) and \( \infty \) does not cross the unit circle centered at \( z = 0 \). Then \( \log(z + 2) \) is analytic inside \( C \) and, by Cauchy theorem, \( \oint_C \log(z + 2) \, dz = 0 \).

(b) \( 1/(z^2 + 1/4) \).

\[ \frac{1}{z^2 + 1/4} = \frac{1}{i(z - i/2)} - \frac{1}{i(z + i/2)} \]

\( z = \pm i/2 \) are the singularities of \( f(z) \) inside the contour. For each summand, we find

\[ \oint_C \frac{1}{i(z - i/2)} \, dz = 2\pi i / i = 2\pi, \quad \oint_C \frac{1}{i(z + i/2)} \, dz = 2\pi i / i = 2\pi, \]

so \( \oint_C f(z) \, dz = 2\pi - 2\pi = 0 \).

Problem #2 (20 points): Evaluate the integral \( \oint_C f(z) \, dz \) where \( C \) is the unit circle centered at the origin for the following \( f(z) \):

(a) \( \frac{e^{iz}}{z} \)

(b) \( \frac{\cos z - 1}{z^3} \)

Solution: Here the only singular point inside the unit circle is \( z = 0 \). We expand the numerators in Taylor series around zero and use the integration of powers formula:

(a) \[ \frac{e^{iz}}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{k=-1}^{\infty} \frac{(iz)^k}{(k+1)!}, \]

power \( z^{-1} \) corresponds to \( k = -1 \), thus \( \oint_C f(z) \, dz = 2\pi i \).

(b) \[ \frac{\cos z - 1}{z^3} = \frac{1}{z^3} - \frac{1}{2z} + \frac{z}{24} + \cdots, \]

so \( \oint_C f(z) \, dz = -2\pi i / 2 = -i\pi \).

Problem #3 (20 points): Evaluate the integrals \( \oint_C f(z) \, dz \) over a contour \( C \), where \( C \) is the boundary of a square with diagonal opposite corners at \( z = -(1 + i)R \) and \( z = (1 + i)R \), where \( R > a > 0 \), and where \( f(z) \) is given by the following (use Eq. (1.2.19) as necessary):

(a) \[ \frac{e^z}{(z - \frac{\pi i}{4} a)^2} \]

(b) \[ \frac{z^2}{2z + a} \]
Solution:

(a) \[ \frac{e^z}{(z - \frac{\pi i}{4} a)^2} \]

Let \( z_0 = \frac{\pi i}{4} a \), it is inside the square; we expand \( e^z \) in Taylor series around \( z = z_0 \) here,

\[
\frac{e^z}{(z - \frac{\pi i}{4} a)^2} = \frac{e^{z_0} e^{z-z_0}}{(z-z_0)^2} = \frac{e^{z_0}}{(z-z_0)^2} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} = 
\]

\[
= \frac{e^{z_0}}{(z-z_0)^2} \left( 1 + (z-z_0) + \frac{(z-z_0)^2}{2} + \ldots \right) = e^{z_0} \left( \frac{1}{(z-z_0)^2} + \frac{1}{z-z_0} + \frac{1}{2} + \ldots \right),
\]

and the only singular point \( z = z_0 \) is inside the contour. Deforming the contour to a circle around \( z = z_0 \) and using Cauchy theorem, we find

\[
\oint_C \frac{e^z}{(z - \frac{\pi i}{4} a)^2} \, dz = 2\pi i e^{z_0} = 2\pi i e^{\frac{\pi i}{4} a}.
\]

(b) \[ \frac{z^2}{2z + a} \]

\[
\frac{z^2}{2z + a} = \frac{(-a/2 + (z + a/2))^2}{2(z + a/2)} = 
\]

\[
= \frac{a^2}{8(z + a/2)} \frac{a + z + a/2}{2},
\]

and the only singular point \( z = -a/2 \) is inside the contour. Deforming the contour to a circle around \( z = -a/2 \) and using Cauchy theorem, we find

\[
\oint_C \frac{z^2}{2z + a} \, dz = \oint_C \frac{a^2}{8(z + a/2)} \, dz = \frac{\pi i a^2}{4}.
\]

Problem #4 (25 points): Let \( f(z) \) be an entire function, with \( |f(z)| \leq C|z| \) for all \( z \), where \( C \) is a constant. Show that \( f(z) = Az \), where \( A \) is a constant.

Solution: Using the (generalized) Cauchy formula,

\[
f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}^2 \, d\zeta,
\]

where \( C = ||\zeta - z| = R \) is the circle of radius \( R \) around \( z \) in \( \zeta \)-plane. Then

\[
|f'(z)| \leq \frac{1}{2\pi} \oint_C \left| \frac{f(\zeta)}{|\zeta - z|^2} \right| |d\zeta| \leq 
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{C(|z| + R)}{R^2} Rd\theta = C(1 + |z|/R) \rightarrow R \rightarrow \infty C,
\]

so \( f'(z) \) is entire and bounded, so it is constant by Liouville theorem. Let \( f'(z) = A \), then \( f(z) = Az + B \), where \( A \), \( B \) are constants. But, since \( |f(z)| \leq C|z| \) for all \( z \), taking \( |z| \rightarrow 0 \), we get \( B = 0 \). Thus, \( f(z) = Az \) as claimed.

Problem #5 (20 points): Discuss whether the sequence \( \{1/(nz)^2\}_{n=1}^{\infty} \) converges and whether the convergence is uniform for: \( 0 < \alpha < |z| < 1 \). Discuss whether the convergence is uniform if \( \alpha = 0 \).
Solution:
\[
\lim_{n \to \infty} \frac{1}{(nz)^2} = \frac{1}{z^2} \lim_{n \to \infty} \frac{1}{n^2} = 0,
\]
so the sequence converges pointwise for every \( z \). If \( |z| > \alpha > 0 \), then
\[
\left| \frac{1}{(nz)^2} \right| = \frac{1}{n^2|z|^2} \leq \frac{1}{\alpha^2},
\]
which is a convergent numerical sequence. Thus, the convergence is uniform for \( 0 < \alpha < |z| \). However, for \( \alpha = 0 \) convergence is not uniform since \( 1/|z|^2 \) is unbounded above in this case.

**Problem #6 (20 points):** Show that the following series converge uniformly in the given region:

(a) \( \sum_{n=1}^{\infty} z^{2n}, \ 0 \leq |z| < R < 1 \)

(b) \( \sum_{n=1}^{\infty} e^{-2nz}, \ R < |Rez| < 1 \)

Solution:

(a)
\[
\left| \sum_{n=1}^{\infty} z^{2n} \right| \leq \sum_{n=1}^{\infty} |z|^{2n} \leq \sum_{n=1}^{\infty} R^{2n} = \frac{R^2}{1 - R^2},
\]
i.e. the series is bounded above by a convergent numerical series which means uniform convergence by Weierstrass M-test.

(b)
\[
\left| \sum_{n=1}^{\infty} e^{-2nz} \right| \leq \sum_{n=1}^{\infty} |e^{-2nz}| = \sum_{n=1}^{\infty} e^{-2n|Rez|} < \sum_{n=1}^{\infty} e^{-2nR} = \frac{e^{-2R}}{1 - e^{-2R}}
\]
for \( R > 0 \), i.e. the series is bounded above by a convergent numerical series for \( R > 0 \) which means uniform convergence by Weierstrass M-test for \( R > 0 \) (but not for \( R \leq 0 \)).

**Problem #7 (20 points):** Find the radius of convergence of the series \( \sum_{n=0}^{\infty} a_n(z) \) where \( a_n(z) \) is given by:

(a) \( (-z^2)^n \)

(b) \( n^{2n} z^{4n} \)

Solution: We apply the ratio test.

(a) \( (-z^2)^n \),
\[
\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{(-z^2)^n}{(-z^2)^{n+1}} \right| = \frac{1}{|z|^2},
\]
therefore the series converges for \( |z| < 1 \) and radius of convergence \( R = 1 \).

(b) \( n^{2n} z^{4n} \)
\[
\left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{n^{2n} z^{4n}}{(n+1)^{2(n+1)} z^{4(n+1)}} \right| = \frac{1}{(n+1)^2(1 + 1/n)^2 |z|^4} \to 0 \quad n \to \infty,
\]
which shows that \( R = 0 \) (series converges only for \( z = 0 \)).

**Problem #8 (15 points):** Find Taylor series expansions around \( z = 0 \) of the following functions in the given regions:
(a) $\frac{z}{1+z^2}, |z| < 1$

(b) $\frac{\sin z}{z}, 0 < |z| < \infty$

(c) $\frac{e^{z^2}-1-z^2}{z^3}, 0 < |z| < \infty$

**Solution:**

(a) $\frac{z}{1+z^2}, |z| < 1$

$$\frac{z}{1+z^2} = z \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}.$$  

(b) $\frac{\sin z}{z}, 0 < |z| < \infty$

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}. $$

(c) $\frac{e^{z^2}-1-z^2}{z^3}, 0 < |z| < \infty$

$$\frac{e^{z^2}-1-z^2}{z^3} = \frac{1}{z^3} \left( \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} - 1 - z^2 \right) = \sum_{n=2}^{\infty} \frac{z^{2n-3}}{n!} = \sum_{n=2}^{\infty} \frac{z^{2n+1}}{n!} \frac{1}{(n+2)!}. $$

**Problem #9 (20 points):** Use the Taylor series for $(1 + z)^{-1}$ about $z = 0$ to find the Taylor series of $\log(1 + z)$ about $z = 0$ for $|z| < 1$.

**Solution:** The Taylor series for $(1 + z)^{-1}$ is just the geometric series

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n,$$

and we know that it converges uniformly in $|z| < 1$. Since $\int (1 + z)^{-1} \, dz = \log(1 + z) + c$ and since the above series converges uniformly, we can integrate it termwise. Taking the (principal) branch of log such that $\log 1 = 0$, we find

$$\log(1 + z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}.$$

**Problem #10 (20 points):** Find a series representation for $1/(1 + z^2)$ for $|z| > 1$. (Hint: see the discussion and hint of problem 3.2.8)

**Solution:** The Taylor series for $1/(1 + z^2)$ is just the geometric series

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n},$$

and we know that it converges in $|z| < 1$. For $|z| > 1, 1/|z| < 1$, so we have

$$\frac{1}{1+z^2} = \frac{1}{z^2(1+1/z^2)} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2(n+1)}}.$$
Extra-Credit Problem #11 (10 points): In Cauchy’s Integral Formula (Eq. (2.6.1)), take the contour to be a circle of unit radius centered at the origin. Let $\zeta = e^{i\theta}$ to deduce

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z} \, d\theta$$

where $z$ lies inside the circle. Explain why we have

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/\bar{z}} \, d\theta$$

and use $\zeta = 1/\bar{\zeta}$ to show

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \left( \frac{\zeta}{\zeta - z} \pm \frac{\bar{z}}{\bar{\zeta} - \bar{z}} \right) \, d\theta$$

whereupon, using the plus sign

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{1 - |z|^2}{|\zeta - z|^2} \, d\theta$$

(a) Deduce the “Poisson formula” for the real part of $f(z)$: $u(r, \phi) = \text{Re} f(z), z = re^{i\phi}$

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{1 - r^2}{(1 - 2r \cos(\phi - \theta) + r^2)} \, d\theta$$

where $u(\theta) = u(1, \theta)$.

(b) If we use the minus sign in the formula for $f(z)$ above, show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \left[ \frac{1 + r^2 - 2r e^{i(\theta - \phi)}}{(1 - 2r \cos(\phi - \theta) + r^2)} \right] \, d\theta$$

and by taking the imaginary part

$$v(r, \phi) = C + \frac{1}{\pi} \int_0^{2\pi} u(\theta) \frac{r \sin(\phi - \theta)}{(1 - 2r \cos(\phi - \theta) + r^2)} \, d\theta$$

where $C = \frac{1}{2\pi} \int_0^{2\pi} v(1, \theta) d\theta = v(r = 0)$. (This last relationship follows from the Cauchy Integral formula at $z = 0$ — see the first equation in this exercise.)

**Solution:** First formula is due to $d\zeta = ie^{i\theta} \, d\theta = i\zeta \, d\theta$. Since $z$ is inside the unit circle, $\bar{z}$ is also and then $1/\bar{z}$ is outside which yields the second formula by Cauchy theorem. Using that $\zeta\bar{\zeta} = 1$ on the unit circle, the third formulas are obtained by adding/subtracting the first two formulas, respectively. The fourth formula is straightforward (again use $\zeta\bar{\zeta} = 1$). Let $f(z) = u + iv$, $u$ and $v$ real.

(a) Take the real part of the fourth formula: $\text{Re} f(z) = u(r, \phi), \text{Re} f(\zeta) = \text{Re} f(e^{i\theta}) = u(1, \theta) = u(\theta), |\zeta| = r$ and

$$|\zeta - z|^2 = (\zeta - z)(\bar{\zeta} - \bar{z}) = 1 - (\zeta\bar{z} + z\bar{\zeta}) + r^2 = 1 - r(e^{i\theta} e^{-i\phi} + e^{i\phi} e^{-i\theta}) + r^2 = 1 - 2r \cos(\theta - \phi) + r^2.$$  

Thus, Poisson formula is obtained.

(b) We have

$$\frac{\zeta}{\zeta - z} - \frac{\bar{z}}{\bar{\zeta} - \bar{z}} = \frac{1 + r^2 - 2re^{i(\theta - \phi)}}{|\zeta - z|^2} = \frac{1 + r^2 - 2re^{i(\theta - \phi)}}{1 - 2r \cos(\theta - \phi) + r^2},$$

thus,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \left[ \frac{1 + r^2 - 2re^{i(\theta - \phi)}}{1 - 2r \cos(\phi - \theta) + r^2} \right] \, d\theta.$$
Taking the imaginary part of the last formula, we get

\[ v(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} v(1, \theta) \left[ \frac{1 + r^2 - 2r \cos(\phi - \theta)}{1 - 2r \cos(\phi - \theta) + r^2} \right] d\theta + \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \left[ \frac{2r \sin(\phi - \theta)}{1 - 2r \cos(\phi - \theta) + r^2} \right] d\theta = \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} v(1, \theta) d\theta + \frac{1}{\pi} \int_0^{2\pi} u(\theta) \left[ \frac{r \sin(\phi - \theta)}{1 - 2r \cos(\phi - \theta) + r^2} \right] d\theta, \]

which is the last claimed formula.