APPM 4/5560: A Note About Problem 1 on HW 4

The random walk described in class is recurrent if and only if $p = 1/2$.

Let $T_0 = \min\{n \geq 1 : X_n = 0\}$. We want to show that $\mathsf{E}_0[T_0] = \infty$ (null recurrence).

Note that T_0 can take on values in $\{1, 2, 3, \ldots\}$.

State 0 is positive recurrent if

$$\mathsf{E}_0[T_0] < \infty$$

and state 0 is null recurrent if

$$\mathsf{E}_0[T_0] = \infty.$$

So, we want to show that $\mathsf{E}_0[T_0] = \infty$.

We saw in class that state 0 is periodic with period 2. (Actually all states have period 2 because they all communicate.) Thus, we can only return in an even number of steps. We found an expression for $p_{00}^{(n)}$, the probability of returning in n steps, but it is not what is needed here since it includes the possibility of visiting 0 several times along the way and yet, here, we are talking about <u>first return times</u>. Our notation for this would be

$$g_0^{(n)} = P_0(T_0 = n).$$

Once you can compute this, you have that

$$\mathsf{E}_0[T_0] = \sum_{n=1}^{\infty} n \cdot P_0(T_0 = n) \sum_{n=1}^{\infty} n \cdot g_0^{(n)}.$$

Finding that probability is trickier than I thought it would be although we do know that it will also be zero for an odd exponent. I will show in this document that

$$\boxed{g_0^{(2n)} = \frac{1}{2n-1} \binom{2n}{n} \frac{1}{2^{2n}}.} \tag{1}$$

FEEL FREE TO USE THIS RESULT WITHOUT READING FURTHER!

1 A Few Brute Force Cases

Consider $P_0(T_0 = 2)$ which is the probability that the first return is in 2 steps.

$$\begin{aligned}
P_0(T_0 = 2) &= P(\text{up on 1st time step and down on 2nd}) + P(\text{down on 1st time step and up on 2nd}) \\
&\overset{indep}{=} P(\text{up on 1st time step}) \cdot P(\text{down on 2nd}) + P(\text{down on 1st time step}) \cdot P(\text{up on 2nd}) \\
&= p \cdot (1-p) + (1-p) \cdot p \\
&= 2p(1-p).
\end{aligned}$$

It is easy to check that this matches (1) for $n = 1$ (and hence $2n = 2$).

Next, we'll look at $P_0(T_0 = 4)$. Let's use a sequence of U's and D's to denote ups and downs. We must have a total of 2 U's and 2 D's in our sequence.

Possible return to 0 moves, and there are $\binom{4}{2} = 6$ include

$$\begin{array}{cccc} U & U & D & D \\ U & D & U & D \\ U & D & D & U \\ D & U & U & D \\ D & U & D & U \\ D & D & U & U \end{array}$$

We can not (for example) go UDUD because, while it would return us to 0 in 4 time steps, this would not be the first return time. From the 6 possibilities, we see that

$$P_0(T_0 = 4) \quad = \quad P(UUDD) + P(DDUU)$$

each has probability $p^2(1-p)^2$ so

$$P_0(T_0 = 4) = 2p^2(1-p)^2.$$

Again, it is easy to check that this matches (1) for $n = 2$ (and hence $2n = 4$).

Now it's getting tougher. To find $P_0(T_0 = 6)$, we have $\binom{6}{3} = 20$ possibilities (combinations of 3 U's and 3 D's) to consider. Most can be ruled out pretty easily. For example, we can not have

$$UDUDUD$$

because we will have returned to zero for the first time in 2 steps and not 6. Still, I think we're approaching the end of where we're going to want to write out all cases...

2 The General Case

Let's write the position of the random walk at time n as the sum

$$S_n := X_1 + X_2 + \cdots + X_n$$

where

$$X_k = \begin{cases} +1 & , \text{ with prob } p \\ -1 & , \text{ with prob } 1-p \end{cases}$$

for $k = 1, 2, \ldots, n$.

We can write the probability that, starting from 0, we first return to 0 at time $2n$ as

$$g_0^{(2n)} = P_0(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n-1} \neq 0, S_{2n} = 0).$$

(Note that, starting from 0, we can never have $S_m = 0$ for odd m so we can remove a lot of those terms but it will be easier to leave them in.)

Let A, B, and C be the events

$$\begin{aligned} A &:= \{S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n} = 0\} \\ B &:= \{S_1 \neq 0, S_4 \neq 0, \ldots, S_{2n-1} \neq 0\} \\ C &:= \{S_2 \neq 0, S_4 \neq 0, \ldots, S_{2n-1} \neq 0, S_{2n} \neq 0\} \end{aligned}$$

Note that $B = A \cup C$ and that A and C are disjoint.

So, we have that

$$P(B) = P(A) + P(C)$$

and so

$$P(A) = P(B) - P(C).$$

That is, in the notation of the problem, we have

$$\begin{aligned} g_0^{(2n)} &= P_0(S_1 \neq 0, S_2 \neq 0, S_{2n-1} \neq 0, S_{2n} = 0) \\ &= P_0(S_1 \neq 0, S_1 \neq 0, \ldots, S_{2n-1} = 0) - P_0(S_2 \neq 0, S_4 \neq 0, \ldots, S_{2n-1} = 0, S_{2n} \neq 0) \end{aligned}$$

In order to continue, we will need the following.

Claim 0: For the random walk with $p = 1/2$,

$$P_0(S_1 \neq 0, S_2 \neq 0, \cdots, S_{2n} \neq 0) = P_0(S_{2n} = 0).$$

I will prove this at the end of this document!

Recall from class that

$$P_0(S_{2n} = 0) = \binom{2n}{n} p^n (1-p)^{2n-n} \stackrel{p=1/2}{=} \binom{2n}{n} \frac{1}{2^{2n}}.$$

It is easy (routine manipulation of factorials) to show that

$$P_0(S_{2n-2} = 0) = P_0(S_{2(n-1)} = 0) = \frac{2n}{2n-1} \cdot P(S_{2n} = 0).$$

By Claim 0, we now have

$$\begin{aligned} g_0^{(2n)} &= P_0(S_{2n-2} = 0) - P_0(S_{2n} = 0) \\ &= \left(\tfrac{2n}{2n-1} - 1\right) P_0(S_{2n} = 0) \\ &= \tfrac{1}{2n-1} P_0(S_{2n} = 0) \\ &= \tfrac{1}{2n-1} \binom{2n}{n} \tfrac{1}{2^{2n}} \end{aligned}$$

as claimed.

3 Proof of Claim 0

We'll need a bunch of other things first.

Claim 1: For the random walk with $p = 1/2$,

$$P_0(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n-1} \geq 0) = 2 \cdot P_0(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0)$$

Proof of Claim 1:

Let D_{2n-1} be the event

$$D_{2n-1} = \{S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n-1} \geq 0\}$$

and let E_{2n} be the event

$$E_{2n} = \{S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0\}.$$

Consider any path in the event E_{2n}. Note that, since $S_1 = X_1$ and since $X_1 \in \{-1, 1\}$, we must have $X_1 = 1$. Draw this on a time versus state space graph. (Sorry, I don't have time to include figures in this document.) Now shift the origin to $(1, 1)$ and the remainder of the E_{2n} path will correspond to a path of length $2n-1$, starting at 0, where all sums are greater than or equal to 0.

Let $|D_{2n-1}|$ and $|E_{2n}|$ be the number of possible paths contained in the events D_{2n-1} and E_{2n}, respectively. Note that $|D_{2n-1}| = |E_{2n}|$ since there is a one-to-one correspondence between paths in each set.

Since any particular path of length $2n-1$ will happen with probability $(1/2)^{2n-1}$

$$P_0(D_{2n-1}) = \frac{|D_{2n-1}|}{2^{2n-1}} = 2\frac{|D_{2n-1}|}{2^{2n}} = 2\frac{|E_{2n}|}{2^{2n}} = 2 \cdot P_0(E_{2n}),$$

as desired. □

Claim 2: For the random walk with $p = 1/2$,

$$P_0(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n} \geq 0) = 2 \cdot P_0(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0)$$

Proof of Claim 2: We will continue to use the set/event notation established in the proof of Claim 1.

We want to show that $P_0(D_{2n-1}) = P_0(E_{2n})$.

Consider the event $D_{2n-1} = \{S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n-1} \geq 0\}$.

Since $2n-1$ is odd and we can only be at 0 at even times, we necessarily have $S_{2n-1} > 0$ for paths in D_{2n-1}.

There are only two possible continuations of this path for one more time step (to time $2n$). We can go one unit up or one unit down. So, the lowest we can possibly be at time $2n$ is 0 and we will definitely land in D_{2n}.

Thus,

$$|D_{2n}| = 2 \cdot |D_{2n-1}|$$

and we have that

$$P(D_{2n}) = 2 \cdot P(D_{2n-1}) = 2 \cdot P(E_{2n})$$

by Claim 1. □

Claim 3: For the random walk with $p = 1/2$,

$$P_0(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = 2 \cdot P_0(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0)$$

Proof of Claim 3:

$$P_0(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) = P_0(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0) + P_0(S_1 < 0, S_2 < 0, \ldots, S_{2n} < 0)$$

$$= 2 \cdot P_0(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0)$$

by symmetry of this random walk about 0 since $p = 1/2$. □

Claim 4: For the random walk with $p = 1/2$,

$$P_0(S_{2n} = 0) = P_0(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n} \geq 0)$$

Proof of Claim 4:

Let F_{2n} be the event that $\{S_{2n} = 0\}$. We wish to show that

$$P_0(F_{2n}) = P_0(D_{2n})$$

Proof of Claim 4:

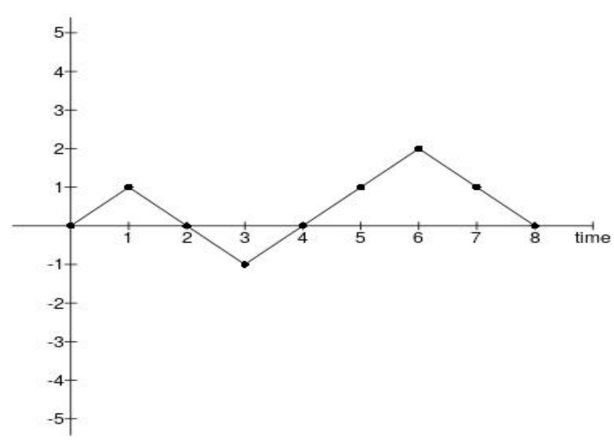
We will show a one-to-one correspondence between paths in F_{2n} and paths in D_{2n}.

Take any path in F_{2n}. All we know is that it ends at state 0 at time $2n$. Along the way, it may or may not have dipped below 0. Let i^* be the minimum value along the way. We know that $i^* \leq 0$.

Case One: the path stayed non-negative all the way. That is, $i^* = 0$. In this case the path is in D_{2n} and so we will make the correspondence directly from the path in F_{2n} to the identical path in D_{2n}.

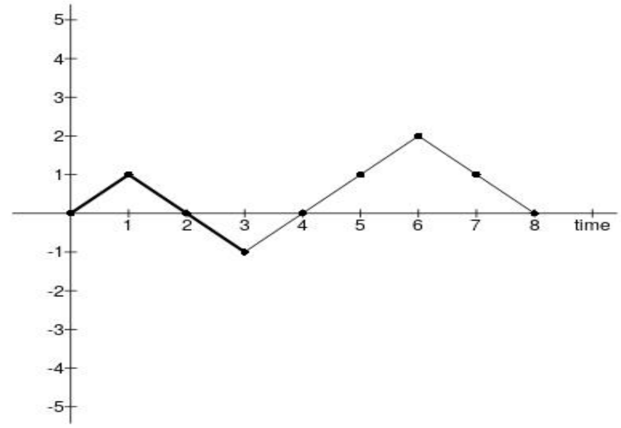
Case Two: the path went negative somewhere. i.e. $i^* < 0$

The path may have achieved the minimum value i^* more than once. Let m^* be the time it <u>first</u> hit i^*.

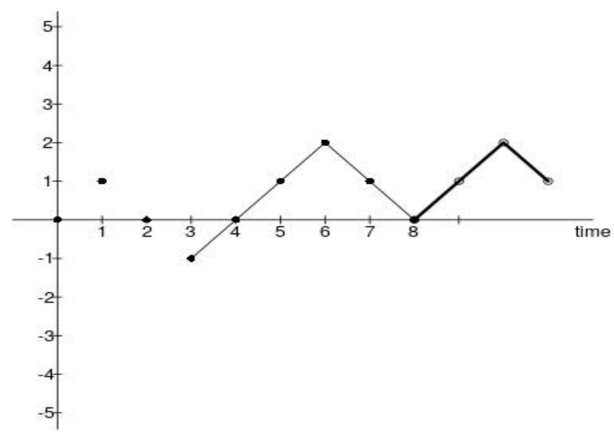


For example, in the Figure above, we have $m^* = 3$ and $i^* = -1$.

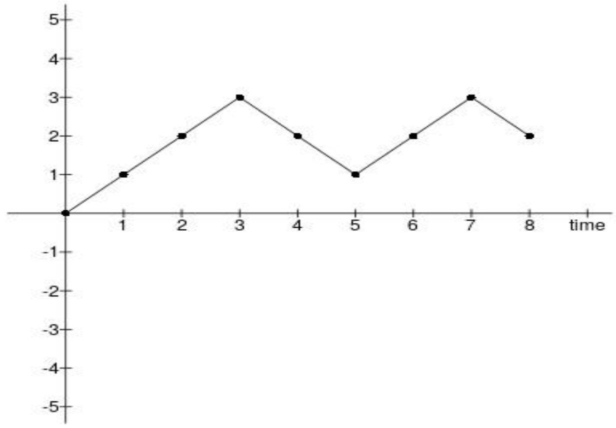
Now, take the part of the path from time 0 to time m^*. Reflect it over the line $t = m^*$, and slide this reflection over so that it starts at the point $(2n, 0)$. For our given example, this means we take this piece (in bold)



and turn it in to this other (bold) piece



Finally, we slide the entire path which now goes from time m^* to time $2n + m^*$ so that it starts at the origin to get



We map the original path in F_{2n} to this path that is in D_{2n}.

We now show that this entire process is invertible. Take any path in D_{2n}. If $S_{2n} = 0$, the path is already in F_{2b}. The path in D_n corresponds to itself in F_{2n}.

If the path in D_{2n} has $S_{2n} > 0$, it necessarily is at some even positive value, say $2i$. Let m be the last time, before $2n$ that the process is at i. The point (m, i) is where the process was attached. Reflect the tail piece from time m to time $2n$ over the horizontal line $t = m$. Slide it to the left m units and down by i units so that the point (m, i) is at the origin.

Finally, slide this entire"Frankenstein process" so that it begins at the origin. The result is a path in F_{2n} such that, if we do the forward procedure as show in the diagrams, we will get back the path in D_{2n}.

So, every path in F_{2n} corresponds to one and only one path in D_{2n} and thus we have that

$$|F_{2n}| = |D_{2n}|$$

and therefore that

$$P(F_{2n}) = \frac{|F_{2n}|}{2^{2n}} = \frac{|D_{2n}|}{2^{2n}} = P(D_{2n}),$$

as desired.

Claim 0: or the random walk with $p = 1/2$,

$$P_0(S_1 \neq 0, S_2 \neq 0, \cdots, S_{2n} \neq 0) = P_0(S_{2n} = 0).$$

Proof of Claim 0:

Claim 1 was used to prove Claim 2. Claims 2, 3, and 4 are then used to prove Claim 0 as follows.

$$\begin{aligned} P_0(S_{2n} = 0) &\overset{Claim\,4}{=} P_0(S_1 \geq 0, S_2 \geq 0, \ldots, S_{2n} \geq 0) \\ &\overset{Claim\,2}{=} 2 \cdot P(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0) \\ &\overset{Claim\,3}{=} P(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0) \end{aligned}$$

□