APPM 4/5560

Problem Set Four (Due Wednesday, February 13th)

1. Just because you will eventually return to a state does not mean it will happen quickly. A recurrent state can be classified as either **positive recurrent** if the expected return time is finite or **null recurrent** if the expected return time is infinite.

   Recall the random walk on the integers with

$$\begin{aligned} p_{i,i+1} &= p \\ p_{i,i-1} &= 1-p \end{aligned}$$

   We saw that this process is recurrent if and only if $p = 1/2$. Show that, in this case, state 0 is null recurrent.

2. (Part of Durrett 1.27) Suppose brands A, B, and C have consumer loyalties of 0.7, 0.8, and 0.9, meaning that a customer who buys (for example) brand A in one week will by it again with probability 0.7 the next week but will switch to one of the other brands with equal probability with probability 0.3.

   What is the limiting market share for each of the 3 products?

3. (Durrett 1.37) An individual has three umbrellas, some at her office, some at home. If she is leaving home in the morning (or leaving work at night) and it is raining, she will take an umbrella if one is there. Otherwise, she gets wet. Assume that, independent of the past, it rains on each trip with probability 0.2. To formulate a Markov chain, let X_n be the number of umbrellas at her current location.

   (a) Find the transition probability matrix for this Markov chain.

   (b) Calculate the limiting fraction of time she gets wet.

4. (Durrett 1.41) **(Reflecting random walk on the line)** Consider the points $1, 2, 3, 4$ to be marked on a straight line. Let $\{X_n\}$ be a Markov chain that moves to the right with probability 2/3 and to the left with probability 1/3 but subject to the rule that if X_n tries to go to the left from 1 or to the right from 4 it stays put.

   (a) Find the transition probability matrix for the chain.

   (b) Find the limiting amount of time the chain spends at each site.

[DON'T FORGET PAGE TWO →]

5. **Simulation** For the following problems, hand in all code.

   Consider a Markov chain on $\{0,1\}$ with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 3/5 & 2/5 \\ 3/4 & 1/4 \end{array} \right\| \end{array}$$

   (a) Find the stationary distribution for this Markov chain analytically. (Not simulation.)

   (b) **Limiting Distribution:** Simulate MANY sample paths of this chain, starting from state 0 forward for "a long time". Report the proportion of time your simulation ends in state 0 and ends in state 1. Do these proportions match your distribution from part (a)?

   (c) **Stationary Distribution:** Draw (simulate) a starting value from the distribution you found in part (a). Simulate forward one time step. Repeat this many times. Report the proportion of time your simulation ends in state 0 and ends in state 1. Do these proportions match your distribution from part (a)?

6. [**Required for 5560 Students Only**] Consider the Markov chain with state space $\{1,2,3,\ldots\}$ and transition probabilities given by

$$\begin{aligned} p_{i,i+1} &= \tfrac{i}{2i+2} && \text{for } i = 1,2,3,\cdots \\ p_{i,i-1} &= \tfrac{1}{2} && \text{for } i = 2,3,4,\cdots \\ p_{i,i} &= \tfrac{1}{2i+2} && \text{for } i = 2,3,4,\cdots \\ p_{11} &= 1 - p_{12} && \end{aligned}$$

   Show that there is no stationary distribution.