APPM 4/5720: Computational Bayesian Statistics, Spring 2018

Problem Set Four (Due Wednesday, May 2nd)

1. Let $X \sim bin(n, \theta)$. Consider the loss function for estimating θ defined as

$$L(\theta, \delta) = \frac{(\delta - \theta)^2}{\theta(1-\theta)}.$$

   Find the Bayes rule/estimator for θ using a $unif(0,1)$ prior.

2. Suppose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Our goal is to estimate σ^2 using frequentist decision theory.

   Consider the statistic $T = t(\vec{X}) = (\sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i)$ and the two decision rules/estimators

$$\delta_1(T) = S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})}{n-1} = \frac{\sum X_i^2 - (\sum X_i)^2/n}{n-1}$$

   and

$$\delta_2(T) = \frac{n-1}{n} S^2.$$

   (Note that $\delta_1(T)$ is the UMVUE and $\delta_2(T)$ is the MLE.)

   Compare the risk functions under squared-error loss. Show that the UMVUE is inadmissible!

3. We have talked quite a bit about the **posterior Bayes estimator** (PBE) for a parameter θ. This is given by the mean of the posterior distribution. Another type of estimator is known as the **maximum a posteriori estimator** (MAP estimator). The MAP estimator for θ is given by the mode of the posterior distribution.

   We saw that, under squared error loss, the Bayes rule/estimator is equal to the PBE. We also saw that, under absolute error loss, it is given by the posterior median. Show, under "0/1 loss":

$$L(\theta, \delta) = \begin{cases} 0 & , \quad \delta = \theta \\ 1 & , \quad \delta \neq \theta \end{cases}$$

   that the Bayes rule/estimator for θ is equal to the MAP estimator for θ.

4. **[Required for 5720 Students Only:]** Consider estimating a k-dimensional θ that lives in a parameter space Ω that is an open subset of $\mathbb{R}^k$. (I usually denote the parameter space as Θ, but in the Bayesian context, we use this to denote a random variable!)

   Let f be a prior for θ. Assume that

   (i) the support of f is equal to Ω, and that

   (ii) $R_\delta(\theta)$ is continuous in θ for all δ in the class of decision functions under consideration.

   Suppose that δ^* is Bayes with finite Bayes risk.

   Prove that δ^* is admissible.