4. Assignment 4

Due Wednesday, February 21

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(1) Suppose that n-by-n matrix A is symmetric and positive definite. Consider the following iteration:

$$A_0 = A$$

$$\textbf{for} \quad k = 1, 2, 3, \ldots$$

$$A_{k-1} = G_k G_k^T$$

$$A_k = G_k^T G_k$$

Here $G_k G_k^T$ is the Cholesky factorization of a symmetric positive definite matrix.

Show that this iteration is well defined, i.e. $G_k^T G_k$ is symmetric and positive definite. Show that if

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

with $a \geq c$ has eigenvalues $\lambda_1 \geq \lambda_2 > 0$, then the A_k converges to $\text{diag}(\lambda_1, \lambda_2)$.

(2) Show that Jacobi's method for finding eigenvalues of a real symmetric matrix is ultimately quadratically convergent.

Assume that all off-diagonal elements of the matrix A_k are $O(\varepsilon)$, where k enumerates Jacobi sweeps. Show that then all rotations of the next Jacobi sweep are of the form

$$\begin{pmatrix} 1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 1 & \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 - O(\varepsilon^2) & \ldots & O(\varepsilon) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & O(\varepsilon) & \ldots & 1 - O(\varepsilon^2) & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 1 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix}.$$

Then demonstrate that this implies that, after the sweep, all off-diagonal elements of A_{k+1} are $O(\varepsilon^2)$. Assume that all eigenvalues are non-zero and distinct.