APPM 4360/5360 **Homework Assignment #3** **Spring 2019**

Problem #1 (10 points):

(a) Given $w_1(z) = (z-2)^{1/3}$,
(i) Where are the branch points of $w_1(z)$; how many Riemann sheets are associated with $w_1(z)$; explain.
(ii) If $z-2 = re^{i\theta}$, $-\pi \le \theta < \pi$, find the branch cut associated with $w_1(z)$; explain.

(b) Given $w_2(z) = \log(z+i)$,
(i) Where are the branch points of $w_2(z)$; how many Riemann sheets are associated with $w_2(z)$; explain.
(ii) If $z+i = re^{i\theta}$, $-\pi/2 \le \theta < 3\pi/2$, find the branch cut associated with $w_2(z)$; explain.

Solution:

(a) (i) $w_1(z)$ is a power function; its branch points are $z=2$ and $z=\infty$; the power is rational $m/l = 1/3$, so $l=3$ and there are three Riemann sheets associated with $w_1(z)$ (or three different branches of it).
(ii) For $-\pi \le \theta < \pi$, the branch cut is on the real axis to the left of $z=2$ i.e. $(-\infty, 2]$. This is where the angle θ is discontinuous and, more importantly, $e^{i\theta/3}$ is also: its values are $e^{i\pi/3}$ and $e^{-i\pi/3}$ at the top and the bottom of the cut, respectively.

(b) (i) $w_2(z)$ is a logarithmic function; its branch points are $z=-i$ and $z=\infty$; there are infinitely many Riemann sheets associated with $w_2(z)$ (or infinitely many different branches of it).
(ii) For $-\pi/2 \le \theta < 3\pi/2$, the branch cut is on the *imaginary* axis down from $z=-i$ i.e. $(-i\infty, -i]$. This is where the angle θ is discontinuous and therefore $\operatorname{Im}\log(z+i) = \theta$ is also: its values are $-\pi/2$ and $3\pi/2$ at the right and the left side of the cut, respectively.

Problem #2 (15 points): Find the branch cut structure associated with the function:

$$f(z) = \log\left(\frac{z-a}{z-b}\right), \qquad a<b,\ a,b \text{ real}$$

where we use the bipolar coordinates:

$$z-a = r_1 e^{i\theta_1}, \quad z-b = r_2 e^{i\theta_2}, \quad \text{with } 0 \le \theta_1 < 2\pi,\ 0 \le \theta_2 < 2\pi$$

Solution:

$$f(z) = \log\left(\frac{z-a}{z-b}\right).$$

This is a log of a rational (single-valued) function. Therefore the branch points are those where

$$\frac{z-a}{z-b} = 0 \qquad \text{or} \qquad \frac{z-a}{z-b} = \infty,$$

i.e. $z=a$ and $z=b$ ($z=\infty$ is not a b.p.). Consider principal angles θ_1, θ_2 s.t.

$$z-a = r_1 e^{i\theta_1}, \qquad z-b = r_2 e^{i\theta_2}, \qquad \Longrightarrow \qquad \log\left(\frac{z-a}{z-b}\right) = \log r + i\Theta = \log\frac{r_1}{r_2} + i(\theta_1 - \theta_2).$$

Then we have (at the top and bottom of x-axis, see pictures in sections 2.2 and 2.3 of the textbook)

θ_1	θ_2	Θ	Region
0	0	0	$\{(x,y)\|x>b, y>0\}$
0	π	$-\pi$	$\{(x,y)\|a<x<b, y>0\}$
π	π	0	$\{(x,y)\|x<a, y>0\}$
π	π	0	$\{(x,y)\|x<a, y<0\}$
2π	π	π	$\{(x,y)\|a<x<b, y<0\}\}$
2π	2π	0	$\{(x,y)\|x>b, y<0\}$

The above table shows that, with these principal angles, we get the branch cut on the interval $[a, b]$.

Problem #3 (20 points): Find the bounded solution to Laplace equation $\nabla^2 T = \partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 = 0$ in the upper half-plane (UHP) $(-\infty < x < \infty,\ y > 0)$, with the boundary conditions given on $y = 0$:

(a)
$$T(x,0) = \{\alpha \text{ on } x < l,\ \beta \text{ on } x > l\}, \qquad \alpha, \beta \text{ are real constants}$$

(b)
$$T(x,0) = \{0 \text{ on } x < l_1,\ \alpha \text{ on } l_1 < x < l_2,\ \beta \text{ on } x > l_2\}, \qquad \alpha, \beta \text{ are real constants}$$

Solution:

(a) We consider the (complex) solutions to Laplace equation in the UHP of the form

$$\Omega(z) = C_1 \log(z-l) + iC_2, \quad z - l = re^{i\theta},$$

with real constants C_1, C_2, so its imaginary part $\psi = C_1\theta + C_2$ yields a bounded solution of Laplace equation. Take e.g. principal branch of log with $0 \le \theta < 2\pi$. We want to identify $T = \psi$, so we apply the boundary conditions. For $x > l$, the angle is $\theta = 0$, and we have $T(x,0) = C_2 = \beta$; for $x < l$, the angle is $\theta = \pi$, so $T(x,0) = C_1\pi + C_2 = \alpha$. Thus, we find the solution $T(x,y)$ satisfying the given BCs:

$$T(x,y) = \frac{\alpha-\beta}{\pi}\theta + \beta = \frac{\alpha-\beta}{\pi}\tan^{-1}\frac{y}{x-l} + \beta.$$

(b) We take the (complex) solutions to Laplace equation in the UHP of the form

$$\Omega(z) = C_1\log(z-l_1) + C_2\log(z-l_2) + iC_3, \qquad z - l_1 = r_1 e^{i\theta_1}, \quad z - l_2 = r_2 e^{i\theta_2},$$

with real constants C_1, C_2, C_3, so its imaginary part $\psi = C_1\theta_1 + C_2\theta_2 + C_3$ yields a bounded solution of Laplace equation. We take principal branches of both logs with $0 \le \theta_1, \theta_2 < 2\pi$. We want to identify $T = \psi$, so we apply the boundary conditions. For $x > l_2$, the angles are $\theta_1 = \theta_2 = 0$, and we have $T(x,0) = C_3 = \beta$; for $l_1 < x < l_2$, the angles are $\theta_1 = 0, \theta_2 = \pi$, so $T(x,0) = C_2\pi + C_3 = \alpha$; for $x < l_1$, the angles are $\theta_1 = \theta_2 = \pi$, so $T(x,0) = (C_1 + C_2)\pi + C_3 = 0$. Thus, we find the solution $T(x,y)$ satisfying the given BCs:

$$\begin{aligned} T(x,y) &= -\frac{\alpha}{\pi}\theta_1 + \frac{\alpha-\beta}{\pi}\theta_2 + \beta = \\ &= -\frac{\alpha}{\pi}\tan^{-1}\frac{y}{x-l_1} + \frac{\alpha-\beta}{\pi}\tan^{-1}\frac{y}{x-l_2} + \beta. \end{aligned}$$

Problem #4 (10 points): From the basic definition of complex integration in section 2.4, evaluate $\oint_C f(z)dz$, where C is the parameterized unit circle enclosing the origin, $C: x(t) = \cos t,\ y(t) = \sin t$ where $f(z)$ is given by:

(a) $f(z) = 1 + z\bar{z}^2$
(b) $f(z) = (z-1)/z$

Solution: We use that $z = e^{it}$, $0 \le t < 2\pi$, so

$$\oint_C f(z)dz = \int_0^{2\pi} f(z(t))z'(t)dt = \int_0^{2\pi} f(e^{it}) \cdot ie^{it}dt.$$

(a) for $f(z) = 1 + z\bar{z}^2$,

$$\oint_C f(z)dz = \int_0^{2\pi} i(1+e^{-it})e^{it}dt = \left(e^{it} + it\right)\Big|_0^{2\pi} = 2\pi i.$$

(b) for $f(z) = (z-1)/z$,

$$\oint_C f(z)dz = \int_0^{2\pi} i(1-e^{-it})e^{it}dt = -2\pi i.$$

Problem #5 (15 points): Evaluate $\oint_C \frac{dz}{z-a}$ where C is the unit circle for a) $|a| < 1$, b) $|a| > 1$; c) What can be said when $|a| = 1$?

Solution: (a) Using the cut contour argument, the contour C can be deformed to a circle C_0 of (small) radius ϵ around a. Then, on C_0, $z - a = \epsilon e^{it}$, $0 \le t < 2\pi$, and

$$\oint_C \frac{dz}{z-a} = \oint_{C_0} \frac{dz}{z-a} = \int_0^{2\pi} \frac{i\epsilon e^{it}dt}{\epsilon e^{it}} = 2\pi i$$

(b) Since the integrand is analytic inside C, by Cauchy theorem, the integral is zero.
(c) If $|a| = 1$, then it is on the contour C, and the integral is undefined as it stands.

Problem #6 (15 points): Consider the integral $\int_0^b (1/z^{1/2})dz$, $b > 0$. Let $z^{1/2}$ have a branch cut along the positive real axis. Show that the value of the integral obtained by integrating along the top half of the cut is exactly minus that obtained by integrating along the bottom half of the cut. What is the difference between taking the principal versus the second branch of $z^{1/2}$?

Solution: Let $z = re^{i\theta}$, then $0 \le \theta < 2\pi$ for the principal branch of $z^{1/2}$. Along the top of the cut, $\theta = 0$, and along the bottom of the cut, $\theta = 2\pi$. Integrating along the top, we get

$$\int_0^b \frac{1}{z^{1/2}}dz = \int_0^b \frac{1}{r^{1/2}}dr = 2b^{1/2},$$

while along the bottom we get

$$\int_0^b \frac{1}{z^{1/2}}dz = \int_0^b \frac{1}{r^{1/2}e^{2i\pi/2}}e^{2i\pi}dr = -2b^{1/2},$$

as was to be shown. Taking the second branch, we have $\theta = 2\pi$ on the top of the cut and $\theta = 4\pi$ on the bottom of the cut. Then, similar calculations give $-2b^{1/2}$ when integrating over the top and $+2b^{1/2}$ when integrating over the bottom.

Problem #7 (15 points):

(a) Evaluate $\oint_C f(z)dz$ using *partial fractions* for $C = \{z(t) = e^{it} : 0 \le t \le 2\pi$ (the unit circle centered at the origin) for the following:

$$f(z) = \frac{1}{(z-1)(z-3)}.$$

(b) Discuss how to evaluate $\int_C \frac{e^z}{z}dz$ where C is a simple closed contour enclosing the origin; explain your reasoning. Hint: use eq. 1.2.19 in the text as necessary.

(c) Evaluate $\int_C \sqrt{z+2}dz$ where C is the unit circle; explain your reasoning.

Solution: Here, we use partial fractions and the fact that

$$\oint_C (z-a)^m dz = \begin{cases} 0, & a \notin D \\ 0, & a \in D,\ m \neq -1 \\ 2\pi i, & a \in D,\ m = -1 \end{cases}$$

where D is the region enclosed by C.

(a) As it stands, the singular point $z = 1$ is on the integration contour, therefore the integral does not exist. If we take instead e.g.

$$f(z) = \frac{1}{(z-1/2)(z-3)} = \frac{1}{2.5(z-3)} - \frac{1}{2.5(z-1/2)},$$

we have that

$$\oint_C f(z)dz = 0 - \frac{2\pi i}{2.5} = -\frac{4\pi i}{5},$$

because the pole at $z = 3$ does not contribute.

(b) We expand e^z in series as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

then

$$\int_C \frac{e^z}{z} dz = \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^{n-1} dz = 2\pi i,$$

where only the term of the sum with $n = 0$ contributes. The others integrate to zero by Cauchy theorem. The $n = 0$ term is evaluated using the deformed contour argument reducing it to the integral over a (small) circle around $z = 0$.

(c) Consider an analytic branch of $\sqrt{z+2}$ such that branch cut joining -2 and ∞ does not cross the unit circle centered at $z = 0$. Then $\sqrt{z+2}$ is analytic inside C and, by Cauchy theorem, $\int_C \sqrt{z+2}dz = 0$.

Extra-Credit Problem #8 (10 points): Consider $I_R = \int_{C_R} \frac{e^{iz}}{z^2} dz$ where C_R is the semicircle of radius R in the upper half-plane with the endpoints $(-R, 0)$ and $(R, 0)$ (C_R is open, it does not include the x-axis). Show that $\lim_{R\to\infty} I_R = 0$.

Solution: We have $C_R = \{z = Re^{i\theta}, 0 \le \theta \le \pi\}$ and use the inequalities

$$|I_R| = \left|\int_{C_R} \frac{e^{iz}}{z^2} dz\right| \le \int_{C_R} \frac{|e^{iz}|}{|z|^2} |dz| = \int_0^{\pi} \frac{|e^{iR(\cos\theta + i\sin\theta)}|}{R^2} |Rie^{i\theta}| d\theta =$$

$$= \int_0^{\pi} \frac{e^{-R\sin\theta}}{R} d\theta \le \frac{1}{R}\int_0^{\pi} d\theta = \frac{\pi}{R} \to_{R\to\infty} 0,$$

which proves that $\lim_{R\to\infty} I_R = 0$.
