APPM 4/5560

Problem Set Three (Due Wednesday, February 6th)

1. Consider the following transition matrices. Identify the communication classes for each. Label each class as transient or recurrent and give a brief justification for your labels.

$$\mathbf{P}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\Vert \begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0.2 & 0 & 0.8 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0.3 & 0 & 0 & 0 & 0.7 \end{array} \right\Vert \end{array} \qquad \mathbf{P}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left\Vert \begin{array}{cccccc} 2/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \end{array} \right\Vert \end{array}$$

2. A two state Markov chain has transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\Vert \begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array} \right\Vert \end{array}$$

Find the first return distribution $g_0^{(n)}$ where

$$g_0^{(n)} := P(\underline{\text{first}} \text{ return to state 0 in } n \text{ steps} \,|\, X_0 = 0)$$

3. Consider the simple two state Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\Vert \begin{array}{cc} 1/2 & 1/2 \\ 0 & 1 \end{array} \right\Vert \end{array}$$

   (a) Find the matrix $\mathbf{P}^n$.

   (b) Find $\lim_{n\to\infty} \mathbf{P}^n$.

   (c) From part (b) classify each state as transient or recurrent. Explain your reasoning.

4. Show that if state i is recurrent and state i does not communicate with state j, then $p_{ij} = 0$. This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a *closed* class.

5. (Parts of Durrett 1.74 and 1.75)

   (a) Consider the *aging chain* on $S = \{0, 1, 2, \ldots\}$ in which, for any age $i \in S$ the individual, over one time step, either gets one day older with probability p_i or dies and returns to age 0 with probability $1 - p_i$.

   Find conditions that guarantee that state 0 is recurrent.

   (b) The opposite of the aging chain is the *renewal chain* with state space $\{0, 1, 2, \ldots\}$ in which $p_{i,i-1} = 1$ when $i > 0$ and $p_{0i} = p_i$ for defined probabilities p_i for $i = 1, 2, \ldots$.

   Show that this chain is always recurrent.

6. **[Required for 5560 Students Only]** Let $T_i = \min\{n \geq 1 : X_n = i\}$. Prove that, if

$$P_i(T_j < \infty) > 0 \qquad \text{and} \qquad P_j(T_i < \infty) < 1$$

then i is transient.