APPM 5470, Methods of Applied Mathematics: Partial Differential Equations and Integral Equations, HW 3, due 10/6 *Fall 2017*

Book Problems:

Chapter 4: 5, 8, 10

Additional Problems:

A1 Consider the moving initial/boundary value problem (IBVP) for the wave equation with unit wave speed

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad x > \gamma(t), \quad t > 0, \\ u(x,0) &= f(x), \quad x > 0, \\ u_t(x,0) &= g(x), \quad x > 0, \\ u(\gamma(t),t) &= h(t), \quad t > 0. \end{aligned}$$

Assume that $f \in C^2(\gamma(0),\infty)$, $g \in C^1(\gamma(0),\infty)$, $h \in C^2(0,\infty)$, and $\gamma \in C^1(0,\infty)$. Also, assume $\gamma(0) = 0$.

(a) Why is the IBVP with $\dot{\gamma}(t) > 1$ for some $t > 0$ ill-posed?

(b) Why is the IBVP with $\dot{\gamma}(t) < -1$ for some $t > 0$ ill-posed?

(c) Suppose $u(x,t)$ is compactly supported in x. Determine the evolution of the total kinetic and potential energy $E = \frac{1}{2}\int_{\gamma(t)}^{\infty}(u_t^2 + u_x^2)\,\mathrm{d}x$. Is this quantity conserved?

(d) Solve the IBVP using the divergence theorem for the vector field $(u_x, -u_t)$ on the domain of dependence for an arbitrary point (x,t), $t > 0$, $x > \gamma(t)$. Check your result by taking $\gamma(t) = 0$ and comparing with the result from class.

(e) Suppose the boundary moves with a constant speed and emits a signal with frequency $\omega > 0$: $\gamma(t) = st$, $|s| < 1$, $f(x) = g(x) = 0$, and $h(t) = \cos(\omega t)$. Find the solution to the IBVP. What is the signal frequency received by an observer standing at the location x_0 where $st < x_0 < t$? The change in frequency is known as the Doppler shift.

A2 Prove that $C^2(\mathbb{R} \times [0,\infty))$ solutions of the initial value problem for the damped wave equation with boundary with damping constant $\mu > 0$

$$\begin{aligned} u_{tt} + \mu u_t - c^2 u_{xx} &= 0, \quad x > 0, \quad t > 0 \\ u(x,0) &= f(x), \quad x > 0 \\ u_t(x,0) &= g(x), \quad x > 0 \\ u(0,t) &= h(t), \quad t > 0 \end{aligned}$$

has a unique solution. *Note: You are not asked to solve this problem!*