APPM 5600: Homework #3
Due in class Wednesday October 4

1 Consider the iteration $x_{k+1} = g(x_k)$ with

$$g(x) = x^2 + x - \frac{1}{4}$$

Find the fixed points, state which are convergent, and give the order of convergence (e.g. linear, quadratic, etc). Justify your answers.

2 Consider the fixed-point iterations

$$x_{k+1} = g(x_k) \text{ and}$$

$$x_{k+1} = G(x_k) = g(g(x_k)) - \frac{(g(g(x_k)) - g(x_k))^2}{g(g(x_k)) - 2g(x_k) + x_k}.$$

The second method (G) is Steffensen's method – repeated use of Aitken extrapolation.

(a) Show that if $G(\alpha) = \alpha$ then $g(\alpha) = \alpha$.

(b) Show that if $g(\alpha) = \alpha$ and $g'(\alpha)$ exists and is not 1, then $G(\alpha) = \alpha$.

(c) Consider the iterative function $g(x) = x^2 + x - 10^{-2}$, with fixed points $\pm 10^{-1}$. For each fixed point, state whether the iterations (using g and G) converge, and give the rate. Justify your answers.

3 The function

$$f(x) = \begin{cases} 0 & x = 1 \\ e^{-\frac{1}{(1-x)^2}} & x \neq 1 \end{cases}$$

is C^∞ and has a single root at $x = 1$. The root has infinite multiplicity. Apply a root finding method of your choice to this function (i.e. write some code). Describe and explain the behavior.

4 Consider a discrete dynamical system of the form $x_{k+1} = g(x_k)$ where g is continuously differentiable and maps the interval $[a, b]$ to itself. Suppose that there is an equilibrium $\alpha = g(\alpha)$, but that it is unstable $g'(\alpha) > 1$. You can't compute this fixed point by forward iterations, but you want to know precisely where it is.

(a) Describe a method to compute the unstable equilibrium (you may assume that you have a 'close enough' guess as to its location).

(b) Apply your method to compute the unstable fixed point of the map $x_{k+1} = \sin^2(\pi x_k)$ on the interval $[0, 1]$ (here $\sin^2$ means $(\sin(\cdot))^2$ not $\sin(\sin(\cdot))$). Give your answer to 6 digits.