Problem #1 (15 points): Let $f(z)$ be a continuous function for all z. Show that if $f(z_0) \neq 0$, then there must be a neighborhood of z_0 in which $f(z) \neq 0$. (Hint: use the reverse triangle inequality: $|a-b| \geq ||a|-|b||$.)

Solution: Proof by contradiction: suppose there is no neighborhood of z_0 in which $f(z) \neq 0$, i.e. in every neighborhood of z_0, no matter how small, there is a point $z_1 \neq z_0$ such that $f(z_1) = 0$. But, by continuity of f, for any $\epsilon > 0$, there is a neighborhood of z_0, $|z - z_0| < \delta$, such that $|f(z) - f(z_0)| < \epsilon$ for all z there. Take ϵ such that $0 < \epsilon < |f(z_0)|$ and consider a point z_1: $|f(z_1) - f(z_0)| < \epsilon$ and $f(z_1) = 0$. We have

$$\epsilon > |f(z_1) - f(z_0)| = |f(z_0)|,$$

which is a contradiction. This proves the statement.

Problem #2 (30 points):

(a) (5 points) Discuss the mapping of the upper half of z-plane for $f(z) = \overline{f(z)}$

(b) (5 points) Discuss the mapping of the first quadrant in the z-plane for $f(z) = 1/z^2$

(c) (20 points) Using the stereographic projection discussed in class which maps the z-plane to the sphere whose center is at $(0,0,1)$, south pole is the origin and north pole is $(0,0,2)$,

find the points on the sphere which correspond to the complex numbers (i) $z = 1+i$; (ii) $z = x$; x real; (iii) $z_0 = x + iy$ where $x; y$ lie on the circle $x^2 + y^2 = r^2$; what happens when $r \to \infty$? (iv) On the other hand, find the numbers in the complex plane which correspond to the following points on the sphere $(X;Y;Z) = (X;Y;Z = 1)$.

Solution:

(a) Since $f(z) = \overline{f(z)}$, $f(z)$ is real for all z, so it maps the upper half of z-plane to (a subset of) the real line.

(b) The boundaries of the first quadrant are mapped to the boundaries of its image. E.g. $[0, +\infty)$ is mapped onto itself (for real z, $1/z^2$ is real; $f(0) = \infty$, $f(\infty) = 0$); for $z = iy$, $0 < y < +\infty$, one has $f(z) = 1/z^2 = -1/y^2$ so it maps to $(-\infty, 0)$. Thus, the boundary of the image is the whole $\mathbb{R}$. A point $z = x + iy$, $x > 0$, $y > 0$, is mapped to $1/(x+iy)^2 = (x-iy)^2/(x^2+y^2)^2 = (x^2 - y^2 - 2ixy)/(x^2+y^2)^2$, which has negative imaginary part. Thus, the first quadrant is mapped onto the *lower* half of $\mathbb{C}$.

(c) (i) $z = 1+i$; i.e. $x = y = 1$. Then the point on the sphere is (X,Y,Z) where

$$X = \frac{4x}{|z|^2+4} = \frac{2}{3}, \qquad Y = \frac{4y}{|z|^2+4} = \frac{2}{3}, \qquad Z = \frac{2|z|^2}{|z|^2+4} = \frac{2}{3}.$$

(ii) $z = x$; x real. Then

$$X = \frac{4x}{|z|^2+4} = \frac{4x}{x^2+4}, \qquad Y = \frac{4y}{|z|^2+4} = 0, \qquad Z = \frac{2|z|^2}{|z|^2+4} = \frac{2x^2}{x^2+4}.$$

All these points are on the large circle – the intersection of the X,Z-plane and the sphere.
(iii) $z_0 = x + iy$ where $x; y$ lie on the circle $x^2 + y^2 = r^2$; what happens when $r \to \infty$? Then

$$X = \frac{4x}{|z|^2+4} = \frac{4x}{r^2+4}, \qquad Y = \frac{4y}{|z|^2+4} = \frac{4y}{r^2+4}, \qquad Z = \frac{2|z|^2}{|z|^2+4} = \frac{2r^2}{r^2+4}.$$

Point (X,Y,Z) lies on a smaller circle in a plane parallel to the z-plane. As $r \to \infty$, $X \to 0$, $Y \to 0$ and $Z \to 2$, which is the north pole as it should be.

(iv) On the other hand, find the numbers in the complex plane which correspond to the following points on the sphere $(X;Y;Z) = (X;Y;Z=1)$. Then $z = x + iy$ where

$$x = \frac{2X}{2-Z} = 2X, \qquad y = \frac{2Y}{2-Z} = 2Y,$$

and $X^2 + Y^2 + (Z-1)^2 = 1$, so $X^2 + Y^2 = 1$. Thus, $x^2 + y^2 = 4$, the circle of radius 2 with the center at the origin in z-plane.

Problem #3 (10 points): Verify if the function $f(x,y) = \sin x \cosh y + i \cos x \sinh y$ satisfies the Cauchy-Riemann conditions. If it does, find the associated analytic function $f(z)$.

Solution: Let $f(x,y) = u(x,y) + iv(x,y)$ where u and v are real. Then $u = \sin x \cosh y$ and $v = \cos x \sinh y$ s.t.

$$u_x = \cos x \cosh y = v_y, \qquad v_x = -\sin x \sinh y = -u_y,$$

i.e. CR conditions hold.

$$f(z) = \frac{(e^{ix} - e^{-ix})(e^y + e^{-y})}{4i} + i\frac{(e^{ix} + e^{-ix})(e^y - e^{-y})}{4} =$$

$$= i\frac{e^{-ix}e^y - e^{ix}e^{-y}}{2} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \sin z.$$

Problem #4 (20 points): Given the imaginary part, $v(x,y)$, of an analytic function, $f(z) = u(x,y) + iv(x,y)$, find the real part, $u(x,y)$, and the analytic function.

(a) $v(x,y) = 3x^2y - y^3 + k$, where k is constant.
(b) $v(x,y) = \frac{-x}{x^2+y^2}$.

Solution:

(a) $v(x,y) = 3x^2y - y^3 + k$, where k is constant.

$$v_x = 6xy = -u_y \implies u = -3xy^2 + h(x),$$

$$v_y = 3x^2 - 3y^2 = u_x \implies u = x^3 - 3xy^2 + g(y),$$

therefore

$$u(x,y) = x^3 - 3xy^2 + \text{ const.},$$

$$f(x,y) = x^3 - 3xy^2 + \text{ const.} + i(3x^2y - y^3 + k) =$$

$$= (x+iy)^3 + ik + \text{ const.},$$

i.e.

$$f(z) = z^3 + ik + c,$$

c is a real constant.

(b) $v(x,y) = \frac{-x}{x^2+y^2}$, i.e. $v(r,\theta) = -\frac{\cos\theta}{r}$.

$$v_r = \frac{\cos\theta}{r^2} = -\frac{u_\theta}{r} \implies u = -\frac{\sin\theta}{r} + h(r)$$

$$v_\theta = \frac{\sin\theta}{r} = r u_r \implies u = -\frac{\sin\theta}{r} + g(\theta)$$

therefore

$$u = -\frac{\sin\theta}{r} + \text{ const.},$$

$$f = -\frac{\sin\theta}{r} + \text{const.} - i\frac{\cos\theta}{r} =$$

$$= -i\frac{\cos\theta - i\sin\theta}{r} + \text{const.} = -i\frac{e^{-i\theta}}{r} + \text{const.} =$$

$$= -i\frac{\bar{z}}{z\bar{z}} + \text{const.} = -\frac{i}{z} + \text{const.}$$

Problem #5 (15 points): Determine where the following functions are analytic; discuss whether there are any singular points.

(a) $\frac{1}{z^4+1}$.
(b) cosech z.
(c) $e^{\cosh z}$.

Solution:

(a) $\frac{1}{z^4+1}$. It is analytic everywhere except for roots of equation $z^4 + 1 = 0$, which are s.t.

$$z^4 = r^4 e^{4i\theta} = -1 = e^{\pi i + 2\pi i k}$$

$$\Longrightarrow\ r = 1, \quad \theta = \frac{\pi(1+2k)}{4}, k \in \mathbb{Z},$$

i.e. different singular points are

$$z = e^{i\pi/4}, \quad z = e^{3\pi i/4}, \quad z = e^{5\pi i/4}, \quad z = e^{7\pi i/4}.$$

(b) cosech z.

$$\text{cosech}\, z = \frac{1}{\sinh z},$$

a ratio of functions analytic in the whole $\mathbb{C}$, so it is analytic except for points where $\sinh z = 0$, i.e. $z = i\pi k, k \in \mathbb{Z}$.

(c) $\exp(\cosh z)$. It is analytic everywhere in $\mathbb{C}$, being a composition of analytic functions, i.e. entire.

Problem #6 (10 points): Let $f(z)$ be analytic in some domain. Show that $f(z)$ is necessarily a constant if either the function $\overline{f(z)}$ is analytic or $f(z)$ assumes only pure imaginary values in the domain.

Solution: Let $f(z) = u + iv$, where u and v are real. Then $\overline{f(z)} = u - iv$. CR conditions for $f(z)$ are $u_x = v_y$ and $v_x = -u_y$, while CR conditions for $\overline{f(z)}$ are $u_x = -v_y$ and $v_x = u_y$. They are only compatible if $u_x = v_y = v_x = u_y = 0$ i.e. if u and v are constant, so $f(z) =$ const.
If analytic $f(z) = iv$, v real, then $v_x = -u_y = 0$ and $v_y = u_x = 0$ (since $u = 0$), so again $f(z) =$ const.

Problem #7 (10 points): Find the location and explain why they are the branch points for the following functions:

(a) $(z+i)^{1/3}$
(b) $\log\frac{1}{(2z+i)}$

Solution:

(a) Let $z + i = \epsilon e^{i\theta_p}$ which is a circular contour centered at $z = -i$. We have just a power $(1/3)$ function in terms of $\zeta = z + i$, so $z = -i$ and $z = \infty$ are branch points.

(b) $\log\frac{1}{(2z+i)} = -\log(2z+i) = -\log 2 - \log(z + i/2)$. This is a number plus $-\log z$ but with shifted origin. So the branch points are $z = -i/2$ and $z = \infty$.

Problem #8 (10 points): Solve for all values of z:

(a) $7+3e^{2z-i\pi}=4$
(b) $\log\frac{3z}{2z+1}=3i\pi$

Solution:

(a)

$$7+3e^{2z-i\pi}=4 \qquad \Leftrightarrow \qquad e^{2z-i\pi}=-1=e^{i\pi+2\pi in},\ n\in\mathbb{Z},$$

therefore

$$2z-i\pi=i\pi+2\pi in \qquad \Leftrightarrow \qquad z=i\pi m,\ m\in\mathbb{Z}.$$

(b)

$$\log\frac{3z}{2z+1}=3i\pi \qquad \Leftrightarrow \qquad \frac{3z}{2z+1}=e^{3i\pi}=-1,$$

therefore $z=-1/5$.

Problem #9 (15 points): Derive $\coth^{-1}z=\frac{1}{2}\log\frac{z+1}{z-1}$ (Hint: use $w=\coth^{-1}z$). Then find $\frac{d}{dz}\coth^{-1}z$.

Solution: One needs to find $w=f(z)$ such that $z=\coth w$. Then

$$z=\frac{\cosh w}{\sinh w}=\frac{e^w+e^{-w}}{e^w-e^{-w}}.$$

Let $\zeta=e^w$, then $e^{-w}=1/\zeta$. Substituting these into the above equation, we find

$$z(\zeta-1/\zeta)=\zeta+1/\zeta$$

or

$$(1-z)\zeta^2=-(1+z) \qquad \Leftrightarrow \qquad \zeta^2=-\frac{1+z}{1-z},$$

i.e.

$$e^{2w}=\frac{z+1}{z-1} \qquad \Leftrightarrow \qquad w=\frac{1}{2}\log\frac{z+1}{z-1}.$$

Then

$$\frac{d}{dz}\coth^{-1}z=w'(z)=\frac{1}{2}\left(\frac{1}{z+1}-\frac{1}{z-1}\right)=\frac{1}{1-z^2},$$

as in the real case (as should be).

Problem #10 (20 points):

(a) Consider the complex velocity potential $\Omega(z)=k\log(z-z_0)$, where k is real and z_0 is a complex constant. Find the corresponding velocity potential and stream function. Show that the velocity is purely radial relative to the point $z=z_0$, and sketch the flow configuration. Such a flow is called a "source" if $k>0$, and a "sink" if $k<0$. The strength M is defined as the outward rate of flow of fluid, with unit density, across a circle enclosing $z=z_0$: $M=\oint_C V_r\,ds$, where V_r is the radial velocity and ds is the increment of arc length in the direction tangent to the circle C. Show that $M=2\pi k$. (See also Subsection 2.1.2.)

Solution: Let $\Omega(x,y)=\phi(x,y)+i\psi(x,y)$. Since $\log(z-z_0)=\log r+i\theta$, where $r=|z-z_0|$ and θ is the angle between the line connecting z_0 and z and positive x direction. Then the velocity potential $\phi=k\log r$ and the stream function $\psi = k\theta$, where $r = |z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ and $\theta = \tan^{-1} \frac{y - y_0}{x - x_0}$. For the components of the velocity field V we get

$$V_r = \frac{\partial\phi}{\partial r} = \frac{k}{r}, \qquad V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0,$$

so we have only nonzero V_r component which means that the velocity is purely radial relative to the point $z = z_0$ and $\text{sign}(V_r) = \text{sign}(k)$ means it points away from z_0 if $k > 0$. To compute M, let C be a circle of radius R around z_0. Then

$$M = \oint_C V_r \, ds = \int_0^{2\pi} \frac{k}{R} \cdot R d\theta = 2\pi k.$$

The streamlines are rays emanating from $z = z_0$ if $k > 0$ and falling into $z = z_0$ if $k < 0$.

(b) Consider the complex velocity potential $\Omega(z) = -ik\log(z - z_0)$, where k is real. Find the corresponding velocity potential and stream function. Show that the velocity is purely circumferential relative to the point $z = z_0$, being counterclockwise if $k > 0$. Sketch the flow configuration. The strength of this flow, called a point vortex, is defined to be $M = \oint_C V_\theta \, ds$, where V_θ is the velocity in the circumferential direction and ds is the increment of arc length in the direction tangent to the circle C. Show that $M = 2\pi k$. (See also Subsection 2.1.2.)

Solution: Let $\Omega(x, y) = \phi(x, y) + i\psi(x, y)$. Since $\log(z - z_0) = \log|z - z_0| + i\theta$, where θ is the angle between the line connecting z_0 and z and positive x direction. Then the velocity potential $\phi = k\theta$ and the stream function $\psi = -k\log r$, where $r = |z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ and $\theta = \tan^{-1} \frac{y - y_0}{x - x_0}$. For the components of the velocity field V we get

$$V_r = \frac{\partial\phi}{\partial r} = 0, \qquad V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{k}{r},$$

so we have only nonzero V_θ component which means that the velocity is purely circumferential relative to the point $z = z_0$ and $\text{sign}(V_\theta) = \text{sign}(k)$ means it is counterclockwise if $k > 0$. To compute M, let C be a circle of radius R around z_0. Then

$$M = \oint_C V_\theta \, ds = \int_0^{2\pi} \frac{k}{R} \cdot R d\theta = 2\pi k.$$

The streamlines are concentric circles around $z = z_0$.

Problem #11 (15 points): Show that the solution to Laplace equation $\nabla^2 T = \partial^2 T/\partial u^2 + \partial^2 T/\partial v^2 = 0$ in the region $-\infty < u < \infty$, $v > 0$, with the boundary conditions $T(u, 0) = T_0$ if $u > 0$ and $T(u, 0) = -T_0$ if $u < 0$, is given by

$$T(u, v) = T_0\left(1 - \frac{2}{\pi}\tan^{-1}\frac{v}{u}\right).$$

Solution: From the text we have solutions to Laplace's equation,

$$\begin{aligned}\Omega(z) &= A\log w + iB \\ &= A\log(re^{i\theta}) + iB \\ &= A\log r + i\underbrace{(A\theta + B)}_{\psi(\theta)}\end{aligned}$$

and so $\psi(\theta)$ satisfies Laplace's equation where $w = re^{i\theta}$, $r = \sqrt{u^2 + v^2}$ and $\theta = \tan^{-1}(v/u)$. Now, apply the boundary conditions. At $\theta = 0$, we have $\psi(0) = B = T_0$ and at $\psi(\pi) = A\pi + T_0 = -T_0$ and so $A = -2T_0/\pi$.

Therefore,

$$\begin{aligned}T(u,v) &= \psi(u,v)\\ &= A\theta + B\\ &= \frac{-2T_0}{\pi}\tan^{-1}(v/u) + T_0\\ &= T_0\left(1 - \frac{2}{\pi}\tan^{-1}\frac{v}{u}\right)\end{aligned}$$

Extra-Credit Problem #12 (20 points):

(a) The above.

(b) Now we'll use this result to solve Laplace's equation in $|z| < 1$ with the boundary conditions

$$T(r=1,\theta) = \begin{cases} T_0, & 0 < \theta < \pi \\ -T_0, & \pi < \theta < 2\pi \end{cases}.$$

Show that the transformation

$$w = i\left(\frac{1-z}{1+z}\right) \qquad z = \frac{i-w}{i+w}$$

maps

- $|z| \le 1$ to the upper-half w-plane ($w = u + iv$ and $v \ge 0$),
- $r = 1$, $0 < \theta < \pi$ onto $v = 0$, $u < 0$, and
- $r = 1$, $\pi < \theta < 2\pi$ onto $v = 0$, $u > 0$.

(c) Use the result in part (b) and the mapping function to show that the solution of the boundary value problem in the circle is given by

$$\begin{aligned}T(x,y) &= T_0\left[1 - \frac{2}{\pi}\cot^{-1}\left(\frac{2y}{1-(x^2+y^2)}\right)\right]\\ &= T_0\left[1 - \frac{2}{\pi}\tan^{-1}\left(\frac{1-(x^2+y^2)}{2y}\right)\right]\end{aligned}$$

or, in polar coordinates,

$$\begin{aligned}T(r,\theta) &= T_0\left[1 - \frac{2}{\pi}\cot^{-1}\left(\frac{2r\sin\theta}{1-r^2}\right)\right]\\ &= T_0\left[1 - \frac{2}{\pi}\tan^{-1}\left(\frac{1-r^2}{2r\sin\theta}\right)\right].\end{aligned}$$

Solution:

(a) see the previous problem.

(b) One could do this in polar or Cartesian coordinates or staying in $(z,\bar{z})$. We do this in Cartesian.

$$\begin{aligned}w &= i\left(\frac{1-z}{1+z}\right)\\ &= i\left(\frac{1-(x+iy)}{1+(x+iy)}\right)\frac{(1+x)-iy}{(1+x)-iy}\\ &= i\left(\frac{(1-x)(1+x) - iy(1-x) - iy(1+x) - y^2}{(1+x)^2+y^2}\right)\\ &= i\left(\frac{1-x^2-iy-iy-y^2}{(1+x)^2+y^2}\right)\\ &= \frac{2y}{(1+x)^2+y^2} + i\frac{1-(x^2+y^2)}{(1+x)^2+y^2}\end{aligned}$$

For u and v we have

$$u(x,y) = \frac{2y}{(1+x)^2+y^2}$$
$$v(x,y) = \frac{1-(x^2+y^2)}{(1+x)^2+y^2}$$

For $|z| \le 1$ we have $x^2+y^2 \le 1$ and we see clearly that $v \ge 0$ and since $y \in \mathbb{R}$ it follows $u \in \mathbb{R}$.

For $r=1$, $x^2+y^2=1$ and $v(x,y)=0$. Now, using $y = r\sin\theta$ we can say

$$y > 0 \Longleftrightarrow 0 < \theta < \pi, \text{ and}$$
$$y < 0 \Longleftrightarrow \pi < \theta < 2\pi,$$

it is the case that

$$u \in (0,\infty) \Longleftrightarrow 0 < \theta < \pi, \text{ and}$$
$$u \in (-\infty,0) \Longleftrightarrow \pi < \theta < 2\pi,$$

(c) Plug in for u and v from part (b) to see

$$\frac{v}{u} = \frac{\frac{1-(x^2+y^2)}{(1+x)^2+y^2}}{\frac{2y}{(1+x)^2+y^2}} = \frac{1-(x^2+y^2)}{2y} = \frac{1-r^2}{2r\sin\theta}$$

and the result follows.

Problem #13 (30 points): Find the location of the branch points and discuss a branch cut structure associated with the function:

(a) $f(z) = \left(\frac{z}{z+1}\right)^{1/2}$
(b) $f(z) = \log(z^2-9)$

Solution:

(a)

$$f(z) = \left(\frac{z}{z+1}\right)^{1/2}.$$

This is a rational function singular at $z=-1$ but single-valued, taken to the power of $1/2$. Therefore the branch points are those where

$$\frac{z}{z+1} = 0 \qquad \text{or} \qquad \frac{z}{z+1} = \infty,$$

i.e. $z=0$ and $z=-1$ ($z=\infty$ is not a b.p.). A branch cut must connect the two branch points, the simplest one is the interval $[-1,0] \in \mathbb{R}$. To confirm this, consider principal angles θ_1, θ_2 s.t.

$$z = r_1 e^{i\theta_1}, \qquad z+1 = r_2 e^{i\theta_2}, \qquad \Longrightarrow \qquad \left(\frac{z}{z+1}\right)^{1/2} = re^{i\Theta} = \left(\frac{r_1}{r_2}\right)^{1/2} e^{i(\theta_1-\theta_2)/2},$$

and the angle ranges are

$$0 \le \theta_1 \le 2\pi, \qquad 0 \le \theta_2 \le 2\pi.$$

Then we have (at the top and bottom of x-axis, see pictures in sections 2.2 and 2.3 of the textbook)

θ_1	θ_2	Θ	Region
0	0	0	$\{(x,y)\|x>0, y>0\}$
π	0	$\frac{\pi}{2}$	$\{(x,y)\|-1<x<0, y>0\}$
π	π	0	$\{(x,y)\|x<-1, y>0\}$
π	π	0	$\{(x,y)\|x<-1, y<0\}$
π	2π	$-\frac{\pi}{2}$	$\{(x,y)\|-1<x<0, y<0\}\}$
2π	2π	0	$\{(x,y)\|x>0, y<0\}$

(b) $f(z) = \log(z^2-9)$. Here z^2-9 is entire single-valued function so the only branch points are those where $z^2-9=0$ or $z^2-9=\infty$. Thus, there are three branch points, $z=\pm 3$ and $z=\infty$. A branch cut must make sure there is no possibility going around any single of them, in this case it must connect all three points. E.g. consider a cut on real axis $\{z = x | x \in [-3, +\infty)\}$.
Consider principal angles θ_1, θ_2 s.t.

$$z-3 = r_1 e^{i\theta_1}, \qquad z+3 = r_2 e^{i\theta_2}, \qquad \Longrightarrow \qquad \log(z^2-9) = \log r + i\Theta = \log(r_1 r_2) + i(\theta_1+\theta_2),$$

and the angle ranges are

$$0 \le \theta_1 \le 2\pi, \qquad 0 \le \theta_2 \le 2\pi.$$

Then we have (at the top and bottom of x-axis, see pictures in sections 2.2 and 2.3 of the textbook)

θ_1	θ_2	Θ	Region
0	0	0	$\{(x,y)\|x>3, y>0\}$
π	0	π	$\{(x,y)\|-3<x<3, y>0\}$
π	π	2π	$\{(x,y)\|x<-3, y>0\}$
π	π	2π	$\{(x,y)\|x<-3, y<0\}$
π	2π	3π	$\{(x,y)\|-3<x<3, y<0\}\}$
2π	2π	4π	$\{(x,y)\|x>3, y<0\}$

This indeed implies the above branch cut.