APPM 4/5560

Problem Set Two (Due Wednesday, January 30th)

1. (a) Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables each taking the value 1 with probability p and the value -1 with probability $1-p$. For $n = 1, 2, \ldots$, define $Y_n = X_1 + X_2 + \cdots + X_n$. Is $\{Y_n\}$ a Markov chain? If so, write down its state space and transition probability matrix.

   (b) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables taking values on $\{0, 1, 2, \ldots\}$ with probabilities $p_i = P(X_5 = i)$. For $n = 1, 2, \ldots$, define $Y_n = \min(X_1, X_2, \ldots, X_n)$. Is $\{Y_n\}$ a Markov chain? If so, write down its state space and transition probability matrix.

2. (Durrett 1.6, adjusted) A taxicab driver moves between the airport A and two hotels B and C according to the following rules.

   - If he is at the airport, he will be at one of the two hotels next with equal probability.
   - If at a hotel, he returns to the airport with probability 3/4 or goes to the other hotel with probability 1/4.

   (a) Find the transition probability matrix for this Markov chain.

   (b) Suppose the driver begins at the airport at time 0. Find the probability that he is back at the airport at time 2.

   (c) Suppose the driver begins at the airport at time 0. Find the probability that he is not at the airport at time 2.

   (d) Suppose the driver begins at the airport at time 0. Find the probability that he is at hotel B at time 3.

3. Consider the Markov chain on $S = \{0, 1, 2\}$ running according to the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array} \right\| \end{array}$$

   Find $P(X_4 = 2 | X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1, X_0 = 0)$.

   Hint: Begin by writing the desired probability as

$$\frac{P(X_4 = 2 X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1 | X_0 = 0)}{P(X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1 | X_0 = 0)}$$

   Note that we already computed something very similar to the denominator in class in the example with Bart's "moods" in the problem where we defined a second transition probability matrix called $\mathbf{Q}$.

4. (Durrett 1.7, adjusted) Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days is rainy.

   Let the weather on day n be R for rain or S for sun. $\{W_n\}$ is not a Markov chain , but the weather for the last two days, $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$.

[OVER →]

(a) Write down the transition probability matrix for $\{X_n\}$.

(b) What is the probability it will rain on Tuesday given that it rained on Sunday and Monday?

(c) What is the probability that is will rain on Wednesday given that it did not rain on Sunday or Monday?

5. **Required for 5560 students only:** For a Markov chain $\{X_n, n \geq 0\}$ with transition probabilities P_{ij}, consider the conditional probability that $X_n = m$ given that the chain started at time 0 in state i and has not yet entered state r by time n, where r is a specified state not equal to either i or m. We are interested in whether this conditional probability is equal to the n stage transition probability of a Markov chain whose state space does not include r and whose transition probabilities are given by

$$Q_{ij} = \frac{P_{ij}}{1 - P_{ir}}, \qquad i, j \neq r.$$

(Note that these transition probabilities are obtained by deleting the rth row and column of $\mathbf{P}$ and "re-normalizing" the rows so that they sum to 1.)

Either prove the equality

$$P(X_n = m | X_0 = i, X_k \neq r, k = 1, 2, \ldots, n) = Q_{im}^{(n)}$$

or construct a counterexample.