APPM 5600: Homework #2
Due in class Wednesday Sept 27

1 At every iteration of the conjugate-gradients algorithm you have to compute a fixed number of matrix-vector products and vector dot products. Supposing you compute all n iterations, so that in exact arithmetic you would have found the exact solution. How many floating point operations does this cost? Give your answer in terms of asymptotic order, e.g. $\mathcal{O}(n^2)$; don't give an exact cost.

2 At each step of CG an optimal solution is sought within the subspace span$\{\boldsymbol{r}_0, \ldots, \boldsymbol{r}_k\}$ where $\boldsymbol{r}_i$ are the residuals (use $\boldsymbol{x}_0 = 0$). Show that this span is equal to span$\{\boldsymbol{b}, \mathbf{A}\boldsymbol{b}, \ldots, \mathbf{A}^{k-1}\boldsymbol{b}\}$. Any subspace of this form is called a 'Krylov' subspace. What happens if $\mathbf{A}^k\boldsymbol{b}$ is within span$\{\boldsymbol{b}, \mathbf{A}\boldsymbol{b}, \ldots, \mathbf{A}^{k-1}\boldsymbol{b}\}$?

3 At each step of the CG method the function

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\mathbf{A}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b}$$

is minimized within some subspace. Show that the **A**-norm of the error is minimized within the search subspace at each step. The **A**-norm of the error is

$$\|\boldsymbol{x} - \boldsymbol{x}_*\|_A^2 = (\boldsymbol{x} - \boldsymbol{x}_*)^T\mathbf{A}(\boldsymbol{x} - \boldsymbol{x}_*), \text{ where } \mathbf{A}\boldsymbol{x}_* = \boldsymbol{b}.$$

4 Problem 26 from Atkinson, chapter 8. Do not do this by hand. FYI this is a finite-difference discretization of a Poisson equation $\Delta u = f$ (you do not need to know this, and it won't help you solve the problem).

5 Problem 13 from Atkinson, chapter 2.

6 Consider the problem of computing $1/a$ for $a \neq 0$ using only addition, subtraction, and multiplication (i.e. using floating-point operations). If a is represented as a normalized floating-point number, we can more specifically consider computing $1/b$ for $b \in [1/2, 1]$.

(a) Show that $f(x) = b - 1/x$ has the appropriate root, and that Newton's method will locally converge quadratically.

(b) Show that Newton's method for this function can be implemented using only addition, subtraction, and multiplication (no division).

(c) Show that if your initial guess is $x_0 = 3 - 2b$, the iteration will converge to the root. FYI this initial guess is based on approximating the function $h(x) = 1/x$ by a line in the interval $[1/2, 1]$.