APPM 5470, Methods of Applied Mathematics: Partial Differential Equations and Integral Equations, HW 2, due 9/22 *Fall 2017*

Book Problems:

Chapter 3: problem 7 (corrected, see below) and 15 (assume the initial condition $u(x,0) = u_0(x)$).

Additional Problems:

3.7 (From `http://www4.ncsu.edu/~shearer/corrections.pdf`) Use the method of characteristics to solve the initial value problem for $u = u(x,y,t)$ on the domain $-\infty < x, y < \infty$ for small $t > 0$:

$$u_t + yu_x + uu_y = 0,$$
$$u(x,y,0) = x + y.$$

Show that the solution has a singularity as $t \to t^*$ for some $t^* > 0$ and find the value of t^*.

A1 Solve the given initial value problem and determine the values of x and y for which it exists.

(a) $u_x + u^2 u_y = 1$, $u(x,0) = 1$

(b) $u_x + \sqrt{u}u_y = 0$, $u(x,0) = x^2 + 1$

A2 Consider the equation $y^2 u_x + xu_y = \sin(u^2)$.

(a) Describe all the characteristic curves in the x-y plane.

(b) For the solution u of the initial value problem with $u(x,0) = x$, determine the values of u_x, u_y, u_{xx}, u_{xy}, u_{yy} on the x-axis.

A3 Consider the inviscid Burgers' equation

$$u_t + uu_x = 0. \tag{1}$$

(a) Suppose $u(x,0) = \tanh x$. For what values of $t > 0$ does the solution of the quasi-linear PDE remain smooth and single valued? Give an approximate sketch of the characteristics in the x-t plane.

(b) Suppose $u(x,0) = -\tanh x$. For what values of $t > 0$ does the solution of the quasi-linear PDE remain smooth and single valued? Give an approximate sketch of the characteristic curves in the x-t plane.

(c) Suppose

$$u(x,0) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}, \quad x \in \mathbb{R}. \tag{2}$$

Sketch the characteristic curves. Solve the Cauchy problem. Hint: solve the problem in each region $x < 0$, $x \in (0,1)$, and $x > 1$ separately and "paste" the solutions together.

A4 Find the closed form solution $u(x,t)$ of the initial-boundary value problem for the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x, t > 0$$
$$u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad x > 0$$
$$u_x(0,t) = \alpha(t), \quad t \ge 0,$$

where g, h, $\alpha \in C^2$. What additional conditions are required on g, h, and α to ensure that $u \in C^2$ even on the characteristic $x = ct$?