APPM 7400: Probability

Solutions to Problem Set One

1. **First**, we will show that an intersection (finite, countable, or uncountable) of σ-fields is a field.

   Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a collection of σ-fields indexed by some, possible uncountably infinite, index set I.

   (i) $\mathcal{F}_\alpha$ a σ-field $\forall \alpha \in I \Rightarrow \emptyset \in \mathcal{F}_\alpha \; \forall \alpha \in I \Rightarrow \emptyset \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha \; \surd$

   (ii) Take any $A \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. Then $A \in \mathcal{F}_\alpha \; \forall \alpha \in I$. Since each $\mathcal{F}_\alpha$ is a σ-field. $A^c \in \mathcal{F}_\alpha$ $\forall\, \alpha \in I$. Thus, $A^c \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha \; \surd$

   (iii) Let $A_1, A_2, \ldots \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. Then $A_1, A_2, \ldots \in \mathcal{F}_\alpha \; \forall\, \alpha \in I$. Since each $\mathcal{F}_\alpha$ is a σ-field, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_\alpha \; \forall\, \alpha \in I$. Thus, $\bigcup_{n=1}^{\infty} A_n \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha$. $\surd$

   Thus, $\bigcap_{\alpha \in I} \mathcal{F}_\alpha$ is a σ-field.

   Second, we will show that there is at least one σ-field containing $\mathcal{A}$.

   - The power set of all subsets of Ω is a σ-field and must contain $\mathcal{A}$.

   Third, we will show that the intersection of all σ-fields containing $\mathcal{A}$, which we now know is non-empty and is in fact a σ-field, is the smallest σ-field containing $\mathcal{A}$.

   - For ease of notation, let $\cap\mathcal{F}$ denote the intersection of all σ-fields containing $\mathcal{A}$.
   - Suppose that $\mathcal{F}$ is a σ-field containing all the sets in $\mathcal{A}$ and is smaller in the sense that $\mathcal{F} \subseteq \cap\mathcal{F}$.
   - Since $\mathcal{F}$ is a σ-field containing $\mathcal{A}$ it will be included in the intersection! Thus, $\cap\mathcal{F} \subseteq \mathcal{F}$.
   - So, we must have that $\mathcal{F} = \cap\mathcal{F}$.

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2. (i) The empty set is considered finite. Therefore $\emptyset \in \mathcal{F}$. $\surd$

   (ii) Take any $A \in \mathcal{F}$. Then A is either finite or "cofinite" ($\mathbb{N} \setminus A$ is finite).

   Case: A is finite.

   Then $\mathbb{N} \setminus A^c = A$ is finite, so $A^c \in \mathcal{F}$.

   Case: A is cofinite.

   Then $A^c = \mathbb{N} \setminus A$ is finite, so $A^c \in \mathcal{F}$.

   Either way, $A^c \in \mathcal{F}$. $\surd$

(iii) Take any $A, B \in \mathcal{F}$. We can show that either $A \cup B \in \mathcal{F}$ or $A \cap B \in \mathcal{F}$.
Case: A and B are both finite.
Then $A \cup B$ is finite, so $A \cup B \in \mathcal{F}$.
Case: A is finite and B is cofinite. (Or vise-versa.)
Then $A \cap B$ is finite, so $A \cap B \in \mathcal{F}$.
Case: A and B are both cofinite.
Then $\mathbb{N} \setminus (A \cap B) = \mathbb{N} \cap (A^c \cup B^c) = (\mathbb{N} \cap A^c) \cup (\mathbb{N} \cap B^c) = (\mathbb{N} \setminus A) \cup (\mathbb{N} \setminus B)$ is finite.
Hence $A \cap B$ is cofinite, so it is in $\mathcal{F}$. $\surd$

Thus, we have shown that $\mathcal{F}$ is a field. However, it is not a σ-field. For example, we can take $A_n = \{2n\}$. Then each A_n is finite, but

$$\bigcup_{n=1}^{\infty} A_n = \{2, 4, 6, 8, \ldots\}$$

is neither finite nor cofinite!

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3. (a) (i) Each $\mathcal{F}_n$ a field $\Rightarrow \emptyset \in \mathcal{F}_n \ \forall \ n \Rightarrow \emptyset \in \cup_n \mathcal{F}_n$.

(ii) Take any $A \in \cup_n \mathcal{F}_n$. Then $A \in \mathcal{F}_n$ for at least one n. Since that $\mathcal{F}_n$ is a field, it must also contain A^c. So, we have A^c in at least one of the fields and hence in the union.

(iii) Take $A_1, A_2, \ldots, A_n \in \cup_n \mathcal{F}_n$. Then each of these sets is in at least one of the fields comprising the union. i.e., There exist $m_1, m_2, \ldots, m_n$ such that $A_i \in \mathcal{F}_{m_i}$.
Take $m = \max\{m_1, m_2, \ldots, m_n\}$. Since $\mathcal{F}_{m_i} \subseteq \mathcal{F}_m$ for $i = 1, 2, \ldots, n$, we have that every $A_i \in \mathcal{F}_m$.
Since $\mathcal{F}_m$ is a field, we have that $\cup_{i=1}^n A_i \in \mathcal{F}_m \subseteq \cup_{i=1}^{\infty} \mathcal{F}_i$, as desired.

(b) Let $\Omega = [0, 1]$, and consider the σ-fields

$$\mathcal{F}_n = \{\emptyset, \Omega, [0, 1 - 1/n], (1 - 1/n, 1)\}.$$

It is easy to verify that each $\mathcal{F}_n$ is a σ-field and that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$.
Take $A_n = [0, 1 - 1/n]$. Clearly this is in $\mathcal{F}_n$. However, $\cup_n A_n = [0, 1)$. As this is not in any $\mathcal{F}_n$, it can not be in the union.

For another counterexample, let $\Omega = \mathbb{N}$ and consider the increasing sequence of σ-fields $\{\mathcal{F}_n\}$ defined as

$$\mathcal{F}_n = \sigma(\{\{1\}, \{2\}, \ldots, \{n\}\}).$$

For example, $\mathcal{F}_1 = \{\emptyset, \{1\}, \mathbb{N} \setminus \{1\}, \mathbb{N}\}$, $\mathcal{F}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \mathbb{N} \setminus \{1\}, \mathbb{N} \setminus \{2\}, \mathbb{N} \setminus \{1, 2\}, \mathbb{N}\}$, etc...
Take $A_n = \{2n\}$, for $n = 1, 2, \ldots$. Note that each A_n is in $\cup_m \mathcal{F}_m$ because $A_n \in \mathcal{F}_m$ for all large enough m. (Though I did not bother to make the subscripts "nice" so that $A_n \in \mathcal{F}_n$.)
Now $\cup_n A_n = \{2, 4, 6, 8, \ldots\}$, but this is not in $\cup_m \mathcal{F}_m$ because it is not in at least one of the $\mathcal{F}_m$ because every set in $\mathcal{F}_m$ is either finite or cofinite and $\{2, 4, 6, 8, \ldots\}$ is neither!