Numerical Analysis 4660 Applied Mathematics

Assignment 1

Due by Friday, February 2nd at 5:30 pm

In what follows $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$, that is A is a $n \times n$ real matrix and x is an $n-$dimensional column vector. Unless noted, all references are to Burden and Faires, Numerical Analysis, 9th edition.

(1) The goal of this problem is to show that

$$\left\|A^t\right\|_2 = \|A\|_2 \tag{0.1}$$

and that

$$\|A\|_2^2 = \left\|AA^t\right\|_2 = \left\|A^tA\right\|_2 . \tag{0.2}$$

(a) Use inner products to show that for any x

$$\|Ax\|_2^2 \leq \left\|A^tA\right\|_2 \|x\|_2^2$$

and hence,

$$\|A\|_2^2 \leq \left\|A^tA\right\|_2 . \tag{0.3}$$

(b) Use (0.3) and property (v) of a matrix norm to conclude that, for any matrix A,

$$\|A\|_2 \leq \left\|A^t\right\|_2$$

and use this property for A and A^t to conclude (0.1).

(c) Use (0.3), property (v) of a matrix norm, and (0.1) to conclude that $\|A\|_2^2 = \|AA^t\|_2$.

(d) Use (0.1) and part (c) to conclude that $\|A\|_2^2 = \|A^tA\|_2$.

(2) Prove that $\text{rank}\,(A) = \text{rank}\,(A^t)$.

(3) Prove that if A is symmetric, that is $A = A^t$, then all its eigenvalues are real-valued.

(4) Here we explore some properties of a diagonally dominant matrix A (see definition 6.20).

(a) Use definition 6.20 and Theorem 7.11 to show that

$$\|A\|_\infty \leq 2 \max_{1 \leq i \leq n} |a_{ii}| .$$

Check this bound on problem 4c of Exercise Set 7.1.

(b) If A has positive diagonal entries, is symmetric, and at least one of the inequalities in the definition of diagonally dominant is strict then $\langle Ax, x\rangle$ is positive for any $x \neq 0$. Prove this property for a 2×2 matrix A.

(5) Here we explore some properties of a second difference matrix. For this problem, $A \in \mathbb{R}^{11\times 11}$ is the tridiagonal matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \vdots \\ \vdots & -1 & \ddots & \ddots & 0 \\ & & \ddots & 2 & -1 \\ 0 & & \cdots & -1 & 2 \end{bmatrix}$$

(a) For $k = 1, 2, 3$ compute

$$Av^{(k)}$$

where the entries of the vectors $v^{(k)}$ are defined as

$$v_i^{(1)} = 1,\ i = 1, \ldots, 11,$$

$$v_i^{(2)} = 1 + \frac{i}{11},\ i = 1, \ldots, 11,$$

and

$$v_i^{(3)} = \frac{i^2 - i + 1}{121},\ i = 1, \ldots, 11$$

and give an explanation of the result.

(b) Implement the Gauss-Seidel iterative method (Algorithm 7.2) and use it to find the solution u and w of the following two linear systems

$$Au = e^{(1)}$$

and

$$Aw = e^{(11)},$$

where $e_l^{(k)} = \delta_{kl}$. That is, $e^{(1)}$ and $e^{(11)}$ are the first and last vectors of the canonical basis of $\mathbb{R}^n$. Show that $A\left(v^{(1)} - u - w\right) = 0$, where $v^{(1)}$ is defined in part (a). Can you explain this result?

(c) Find the eigenvalues of A and verify that your result matches the property mentioned in problem 4(b).

(d) If $\lambda = \rho(A)$ is the largest eigenvalue of A, compute

$$c_k = \frac{(Av)_k}{\lambda\, v_k},\ k = 1, \ldots, 11$$

where

$$v_k = \left(1 - \sqrt{\lambda}\right)^k,\ k = 1, \ldots, 11.$$

Can you explain this result?