NA 5610.

1. Assignment 1

Due Wednesday, January 31

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(1) Any square matrix may be represented in the form A = SV, where S is a Hermitian nonnegative definite matrix and V is a unitary matrix. This is the so-called polar decomposition of matrices (by analogy with the polar decomposition of complex numbers).

Using the polar decomposition, derive the singular value decomposition for square matrices (do not forget to formulate explicitly what is the singular value decomposition of an arbitrary matrix).

- (2) For each of the following statements, prove that it is true or give an example to show that it is false. In all questions $A \in \mathbb{C}^{n \times n}$.
 - (a) If A is real and λ is an eigenvalue of A, then so is $-\lambda$.
 - (b) If A is real and λ is an eigenvalue of A, then so is $\overline{\lambda}$.
 - (c) If λ is an eigenvalue of A and A is non-singular, then λ^{-1} is an eigenvalue of A^{-1} .
 - (d) If A is Hermitian and λ is an eigenvalue of A, then $|\lambda|$ is a singular value of A.
- (3) A matrix $S \in \mathbb{C}^{m \times m}$ such that $S^* = -S$ is called skew-Hermitian (recall that S^* denotes the adjoint matrix).

Show that

- (a) eigenvalues of *S* are pure imaginary
- (b) matrix I S is non-singular
- (c) matrix $Q = (I S)^{-1}(I + S)$ is unitary (it is known as the Cayley transform of matrix S).
- (4) Given $A \in \mathbb{C}^{n \times n}$, use Schur's decomposition to show that, for every $\varepsilon > 0$, there exists a diagonalizable matrix *B* such that $||A B||_2 \le \varepsilon$. This shows that the set of diagonalizable matrices is dense in $\mathbb{C}^{n \times n}$ and that Jordan canonical form is not a continuous matrix decomposition.