APPM 4/5720: Computational Bayesian Statistics, Spring 2018

Problem Set One (Due Friday, January 26th)

1. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed normal random variables with mean μ and variance σ^2. (We write $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$.)

   Assume that σ^2 is fixed and assume a $N(\mu_0, \sigma_0^2)$ prior distribution for the parameter μ. (Assume that μ_0 and σ_0^2 are known hyperparameters.)

   Find the posterior distribution for μ. (Name it.) Is the normal distribution a conjugate prior for μ for this model? Explain.

2. Suppose that X has a gamma distribution with parameters α and β. We write $X \sim \Gamma(\alpha, \beta)$. In our class, this will always mean that X has the pdf

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \, I_{(0,\infty)}(x).$$

   (Whereas for some people it means that the pdf is $f(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \, I_{(0,\infty)}(x)$.)

   For our parameterization, α is known as a *shape parameter* and β is known as an *inverse scale parameter*. (For the other parameterization, β is known as the *scale parameter*.)

   (a) Suppose that $X \sim \Gamma(\alpha, \beta)$. Find the probability density function for $Y := 1/X$. Although this might not be a "recognizable" distribution for you, we call the distribution the "inverse gamma distribution" and will denote it as $Y \sim IG(\alpha, \beta)$.

   (b) Suppose that $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ with μ fixed and known.

   Assume an inverse gamma prior distribution for σ^2, with hyperparameters α and β. (i.e. that $\sigma^2 \sim IG(\alpha, \beta)$)

   Find the posterior distribution for σ^2. Is the inverse gamma a conjugate prior for this normal model? Explain.

3. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables with rate λ. Find a conjugate prior for λ. (Prove that it is a conjugate prior.)

4. (Gelman et.al., Ch 1, Problem 3) Probability calculation for genetics (from Lindley, 1965): Suppose that in each individual of a large population there is a pair of genes, each of which can be either x or X, that controls eye color: those with xx have blue eyes, while heterozygotes (those with Xx or xX) and those with XX have brown eyes. The proportion of blue-eyed individuals is p^2 and of heterozygotes is $2p(1-p)$, where $0 < p < 1$. Each parent transmits one of its own genes to the child; if a parent is a heterozygote, the probability that it transmits the gene of type X is $\frac{1}{2}$. Assuming random mating, show that among brown-eyed children of brown-eyed parents, the expected proportion of heterozygotes is $2p/(1+2p)$. Suppose Judy, a brown-eyed child of brown-eyed parents, marries a heterozygote, and they have n children, all brown-eyed. Find the posterior probability that Judy is a heterozygote and the probability that her first grandchild has blue eyes.

5. The random variables $X_1, X_2, \ldots, X_n$ are said to be *exchangeable* if the joint distribution of $(X_1, X_2, \ldots, X_n)$ is the same as the joint distribution of $(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})$ for any permutation π of the indices $\{1, 2, \ldots, n\}$.

   (a) Show that if $X_1, X_2, \ldots, X_n$ are iid, then they are exchangeable.

   (b) Give an example, ($n = 2$ is fine) to show that exchangeability does not necessarily imply "iid-ness".