APPM 3570: Homework Set 12

Due Wed. Apr. 18

1. Chapter 6 in Ross: Problems 38, 39, 40, 41, 42, 43; Theoretical Exercises 14, 19

2. **Battery factory.** A factory produces batteries with a variety of possible lifetimes. The rate at which a battery dies is λ, and its lifetime is x. The joint density function for a battery's death rate and lifetime is
$$f(x,\lambda) = \lambda e^{-\lambda} e^{-\lambda x}, \quad x \geq 0, \quad \lambda > 0.$$
(a) Check that $f(x,\lambda)$ is a joint density.
(b) What is the marginal densities $f_\lambda(\lambda)$ and $f_x(x)$? What is the expected lifetime $E[x]$ of a battery?
(c) What is the conditional density $f_{x|\lambda}(x|\lambda)$, the p.d.f. of x given the death rate λ? Given $\lambda = 1/$hour, what is the expected lifetime of a battery?
(d) What is the conditional density $f_{\lambda|x}(\lambda|x)$, the p.d.f. of λ given the lifetime x? Given a lifetime 1hour, what is the expected death rate λ of the battery?

3. **Game show.** You may be familiar with the famous *Monty Hall Problem.* Three doors hide two goats and one car – the goal is to pick the door with the car. A contestant chooses one of three doors and is then shown one of the other doors, which reveals a goat. At this point, they may switch doors (to the remaining unopened door) or not. If they switch doors, their probability of getting the car is 2/3, whereas if they don't switch, the probability of getting the car is 1/3.
(a) Prove the above result by applying Bayes' Rule to the following conditional probability
$$P(C_2|H_3, X_1)$$
the probability the car is behind door 2 (C_2) given that the contestant originally selects door 1 (X_1) and the host opens door 3 (H_3) revealing a goat. You are welcome to look at internet or book references for help.
(b) Two contestants play the game, who don't believe the above result (e.g., look up Scott Smith, PhD). When they play a game, their probability of switching doors is 1/2. Compute the p.m.f. for the number of switches $P(S = s)$, $s = 0, 1, 2$; the conditional p.m.f. for the number of wins given the number of switches $P(W = k|S = s)$, $k = 0, 1, 2$; and then use this to compute the joint mass function describing the probability $P(W = k, S = s)$ of k wins and s switches between the two contestants.
(c) What is the expected number of wins when $S = 1$?
(d) Use Bayes Rule to compute $P(S = 2|W = 2)$. That is, given there were two wins $W = 2$, what is the probability there were $S = 2$ switches?

Extra Credit: Correctly complete this problem to add 5 points back to your score on Exam 2.

4. (a) For $X \sim U(-1, 1)$ and $Y \sim \mathcal{N}(0, 1)$ independent, determine the p.d.f. of $Z = X + Y$.
(b) Show that $f_z(-z) = f_z(z)$.
(c) Use your result from (b) to determine $P(Z > 0)$. You should not have to evaluate any integrals.