APPM 4/5560

Solutions to Problem Set Ten

1. By number of steps to acceptance, I meant the number of runs through the steps of the entire accept-reject algorithm. If you return 3 times this number, that's understandable and you'll get full credit.

   Since the trials are independent, the number of trials until the first acceptance is a geometric random variable (the one that starts from 1) with some parameter p. The expected number of trials is then $1/p$. We need to find p.

$$\begin{aligned}
p &= P(\text{ accept on any one trial }) \\
&= P\left(U \le \tfrac{f(Y)}{ch(Y)}\right) \\
&= \textstyle\int_{-\infty}^{\infty} P\left(U \le \tfrac{f(Y)}{ch(Y)} \Big| Y = y\right)\, h(y)\, dy \\
&= \textstyle\int_{-\infty}^{\infty} P\left(U \le \tfrac{f(y)}{ch(y)} \Big| Y = y\right)\, h(y)\, dy \\
&\overset{indep}{=} \textstyle\int_{-\infty}^{\infty} P\left(U \le \tfrac{f(y)}{ch(y)}\right)\, h(y)\, dy \\
&= \textstyle\int_{-\infty}^{\infty} \tfrac{f(y)}{ch(y)}\, h(y)\, dy
\end{aligned}$$

   since $P(U \le u) = u$ for $0 < u < 1$.

   After cancelling the $h(y)$, we get

$$p = \frac{1}{c} \underbrace{\int_{-\infty}^{\infty} f(y)\, dy}_{1} = \frac{1}{c}$$

   Thus, the expected number of steps to acceptance is

$$\boxed{1/p = c}$$

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2. The $\Gamma(3, 2)$ density is

$$f(x) = 4x^2 e^{-2x}$$

   for $x > 0$

   Let's try an h that is heavier in the tails and easy to sample from

$$h(x) = e^{-x}$$

   for $x > 0$.

   We need to find a c such that $ch(x) \ge f(x)$ for all $x > 0$. That is, we want

$$c \ge \frac{f(x)}{h(x)} \quad \text{for all } x > 0.$$

Using Calculus to maximize the ratio $r(x) := f(x)/h(x)$, we get that the maximum value occurs at $x = 2$ and is $c = f(2)/h(2) = 16e^{-2} \approx 2.165365$. So as to not worry about rounding error, let's use the slightly higher upper bound $c = 2.2$ in our algorithm.

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3. We will first determine the stationary distribution for the number of pairs of players in the system. We will then use this to determine the stationary distribution for the number of courts occupied.

   The birth rates are

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda = 3$$

   with

$$\lambda_4 = \lambda_5 = \cdots = 0.$$

   Since the mean time that a court is occupied is 1, the departure rate for a single court is $\mu = 1/1 = 1$ per hour.

   The system death rates are then $\mu_0 = 0$ and

$$\mu_1 = \mu = 1, \;\; \mu_2 = \mu_3 = \mu_4 = 2\mu = 2.$$

   The 2μ comes from the rate of the exponential system departure time derived from a minimum of 2 exponentials,each with rate μ.

   If π_n is the stationary distribution for this process that counts the number of pairs of players in the system, we know that

$$\pi_5 = \pi_6 = \cdots = 0$$

   since there can not be 5 or more pairs in the system.

   Now

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0 = 3\pi,$$

$$\pi_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0 = \frac{9}{2}\pi_0,$$

$$\pi_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}\pi_0 = \frac{27}{4}\pi_0,$$

   and

$$\pi_4 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4}\pi_0 = \frac{81}{8}\pi_0,$$

   we know that

$$\begin{aligned} 1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 \\ &= \pi_0\left(1 + 3 + \tfrac{9}{2} + \tfrac{27}{4} + \tfrac{81}{8}\right) \end{aligned}$$

   which gives us that $\pi_0 = 8/203$ and therefore that

$$\pi_1 = 3\pi_0 = \frac{24}{203},$$

$$\pi_2 = \frac{9}{2}\pi_0 = \frac{36}{203},$$

$$\pi_3 = \frac{27}{4}\pi_0 = \frac{54}{203},$$

and

$$\pi_4 = \frac{81}{8}\pi_0 = \frac{81}{203}.$$

The stationary distribution for the number of courts occupied is

$$\begin{aligned} \pi_0^* &= \pi_0 = \tfrac{8}{203} \\ \pi_1^* &= \pi_1 = \tfrac{24}{203} \\ \pi_2^* &= \pi_2 + \pi_3 + \pi_4 = \tfrac{171}{203}. \end{aligned}$$

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4. For the M/M/∞ queue, we computed the stationary distribution in class. The long-run probability of there being n customers in the system is

$$\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu}$$

for $n = 0, 1, 2, \ldots$

For this problem, we must first put λ and μ into the same units, say minutes. So, $\lambda = 74/60$ calls per minute. We want to find the minumum value of N such that

$$\sum_{n=0}^{N} \pi_n \geq 0.9999$$

when $\lambda = 74/60$ and $\mu = 1/4.2$.

After checking the sum for $N = 0, 1, 2, \ldots$, we determine that the value of N should be 16.

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5. (a) Let $X(t)$ be the number of people (organisms, things, particles, etc...) in the population at time t. Then $\{X(t)\}$ is a birth-and-death model with parameters

$$\begin{aligned} \lambda_i &= i\lambda + \theta \quad , \quad i = 0, 1, \ldots, N-1 \\ \lambda_i &= i\lambda \quad , \quad i = N, N+1, \ldots \\ \mu_i &= i\mu \quad , \quad i = 1, 2, \ldots \end{aligned}$$

(b) Note that

$$\begin{array}{rcl} \lambda_0 & = & \theta = 1 \\ \lambda_1 & = & \lambda + \theta = 1 + 1 = 2 \\ \lambda_2 & = & 2\lambda + \theta = 2(1) + 1 = 3 \\ \lambda_3 & = & 3\lambda = 3(1) = 3 \\ \lambda_4 & = & 4\lambda = 4(1) = 4 \\ \lambda_5 & = & 5\lambda = 5(1) = 5 \\ \vdots & & \vdots \end{array}$$

and that

$$\mu_i = i\mu = 2i \text{ for } i = 0, 1, 2, \ldots .$$

So, we have

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0 = \frac{1}{2}\pi_0,$$

$$\pi_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0 = \frac{1}{4}\pi_0,$$

$$\pi_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}\pi_0 = \frac{1}{8}\pi_0,$$

$$\pi_4 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4}\pi_0 = \frac{3}{4}\frac{1}{2^4}\pi_0$$

$$\pi_5 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4}{\mu_1\mu_2\mu_3\mu_4\mu_5}\pi_0 = \frac{3}{5}\frac{1}{2^5}\pi_0,$$

$$\pi_6 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\pi_0 = \frac{3}{6}\frac{1}{2^6}\pi_0$$

$$\vdots$$

$$\pi_n = \frac{3}{n}\frac{1}{2^n}\pi_0.$$

To find π_0:

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 3\sum_{n=4}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n \right]. \tag{1}$$

To do the sum, note that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots == -\sum_{n=1}^{\infty} \frac{1}{n}x^n$$

which implies that

$$\sum_{n=1}^{\infty} \frac{1}{n}x^n = -\ln(1-x).$$

Thanks Justin!

So,

$$\begin{array}{rcl} \sum_{n=4}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n & = & \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n - \frac{1}{2} - \frac{1}{2}\frac{1}{4} - \frac{1}{3}\frac{1}{8} \\ & = & -\ln(1/2) - \frac{2}{3} = \ln(2) - \frac{2}{3} \end{array}$$

So (1) becomes

$$1 = \pi_0 \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 3\left(\ln(2) - \frac{2}{3}\right) \right] = \ln 8 - \frac{1}{8}$$

$$\Rightarrow \quad \pi_0 = \left[\ln 8 - \frac{1}{8}\right]^{-1} \approx 0.5116$$

The proportion of time immigration is restricted is

$$\begin{aligned}
\sum_{n=N}^{\infty} \pi_n &= \sum_{n=3}^{\infty} \pi_n \\
&= \left[\ln 8 - \tfrac{1}{8}\right]^{-1} \left[\tfrac{1}{8} + 3\textstyle\sum_{n=4}^{\infty} \tfrac{1}{n} \left(\tfrac{1}{2}\right)^n\right] \\
&= \left[\ln 8 - \tfrac{1}{8}\right]^{-1} \left[\tfrac{1}{8} + 3\left(\ln 2 - \tfrac{2}{3}\right)\right] \approx 0.1046
\end{aligned}$$

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6. Let state 0 be that the machine is working and let state i, for $i = 1, \ldots, k$ be that the machine is in repair stage i.

   Then the generator matrix $\mathbf{Q}$ looks like this

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\mu_1 & \mu_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\mu_2 & \mu_2 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & \cdots & -\mu_{k-1} & \mu_{k-1} \\ \mu_k & 0 & 0 & 0 & 0 & \cdots & 0 & -\mu_k \end{bmatrix}$$

   Now using the fact that if $\vec{\pi} = [\pi_0, \pi_1, \ldots]$ is stationar y if and only if $\vec{\pi}Q = \vec{0}$, we peel off the equations

$$-\lambda\pi_0 + \mu_k\pi_k = 0$$

$$\lambda\pi_0 - \mu_1\pi_1 = 0$$

$$\mu_i\pi_i - \mu_{i+1}\pi_{i+1} = 0, \qquad i = 1, 2, \ldots, k-1.$$

   Combine these in this way:

$$\mu_i\pi_i = \mu_{i-1}\pi_{i-1} = \mu_{i-2}\pi_{i-2} = \cdots = \lambda\pi_0$$

   Therefore

$$\pi_i = \frac{\lambda}{\mu_i}\pi_0,$$

   so

$$\sum_{i=0}^{k} \pi_i = \left[1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i}\right]\pi_0.$$

   So, we have that

$$\pi_0 = \left[1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i}\right]^{-1}$$

   and, for $i = 1, 2, \ldots, k$,

$$\pi_i = \frac{\lambda}{\mu_i}\pi_0.$$

(a)

$$\pi_i = \frac{\lambda}{\mu_i} \left[1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i} \right]^{-1}$$

(b)

$$\pi_0 = \left[1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i} \right]^{-1}$$