APPM 4/5560

Problem Set Ten (Due Wednesday, April 24th)

1. Recall the accept-reject algorithm for simulating from a target density f on $(-\infty, \infty)$. (Assume f is continuous.) One must find a function g such that $g(x) \geq f(x)$ for all x, define a pdf $h(x) = g(x)/c$ where $c = \int_{-\infty}^{\infty} g(x)\,dx$ and then perform the following steps:

   $\boxed{1}$ Simulate $Y \sim h$.

   $\boxed{2}$ Simulate $U \sim unif(0, 1)$.

   $\boxed{3}$ If $U \leq \frac{f(Y)}{g(Y)}$, accept Y as a draw from f. Otherwise, discard everything and return to Step 1.

   Find the expected number of steps until the first acceptance.

2. **[Simulation]** Use the accept-reject algorithm to simulate $100,000$ values from the $\Gamma(3, 2)$ distribution. Turn in your code.

   (a) Make a histogram of your results and overlay the $\Gamma(3, 2)$ density.

   (b) What was the average number of steps until acceptance seen in your $100,000$ trials? How does this compare to the answer to Problem 1?

3. (Durrett 8.22) There are two tennis courts. Pairs of players arrive at a rate of 3 per hour and play for an exponentially distributed amount of time with mean 1 hour. If there are already two pairs of players waiting, new arrivals will leave. Find the stationary distribution for the number of courts occupied.

4. CU is overhauling it's entire telephone system. They would like to know how many phone lines they need to ensure that 99.99% of the time they have sufficient capacity to cover all outgoing calls. After extensive data collection, they estimate that the rate of calls going off campus is 74 per hour and they estimate that the mean call length is 4.2 minutes. Use an M/M/∞ queue to estimate the number of phone lines needed to meet the 99.99% capacity requirement.

5. Each individual in a biological population is assumed to give birth at an exponential rate λ, and to die at an exponential rate μ. In addition, there is an exponential rate of increase θ due to immigration. However, immigration is not allowed when the population size is N or larger.

   (a) Set this up as a birth and death model.

   (b) If $N = 3$, $\lambda = \theta = 1$, and $\mu = 2$, determine the proportion of time that immigration is restricted.

6. **Required for 5560 only:** After being repaired, a machine functions for an exponential time with rate λ and then fails. Upon failure, a repair process begins. The repair process proceeds sequentially through k distinct phases. First a phase 1 repair must be performed, then a phase 2, and so on. The times to complete these phases are independent, with phase i taking an exponential amount of time with rate μ_i, $i = 1, 2, \ldots, k$.

   (a) What proportion of time is the machine undergoing a phase i repair?

   (b) What proportion of time is the machine working?